

THE POLYNOMIAL REPRESENTATION OF THE DOUBLE AFFINE HECKE ALGEBRA OF TYPE (C_n^\vee, C_n) FOR SPECIALIZED PARAMETERS

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ABSTRACT. In this paper, we study the polynomial representation of the double affine Hecke algebra of type (C_n^\vee, C_n) for specialized parameters. Inductively and combinatorially, we give a linear basis of the representation in terms of linear combinations of non-symmetric Koornwinder polynomials. The basis consists of generalized eigenfunctions with respect to q -Dunkl-Cherednik operators \hat{Y}_i , and it gives a way to cancel out poles of non-symmetric Koornwinder polynomials. We examine irreducibility and Y -semisimplicity of the representation for the specialized parameters. For some cases, we give a characterization of the subrepresentations by vanishing conditions for Laurent polynomials.

1. INTRODUCTION

In 1990's, Cherednik introduced the double affine Hecke algebra (DAHA). ([ChBook]). It is a unital associative algebra with some parameters attached to affine root systems, and it contains two affine Hecke algebras. The DAHA has an action on a ring of multivariable Laurent polynomials, which is called a polynomial representation. For reduced affine root systems, he proved irreducibility of the polynomial representation for generic parameters, and solved conjectures (duality relations, evaluation formulas, and so on) on Macdonald polynomials[Ma] by exploiting an anti-involution of the DAHA.

For the type (C_n^\vee, C_n) , which is a non-reduced affine root system, Noumi introduced a polynomial representation of the affine Hecke algebra of type C_n to study Macdonald-Koornwinder polynomials ([No]). His realization of operators is called the Noumi representation. Standing on Noumi's work, Sahi introduced the double affine Hecke algebra of type (C_n^\vee, C_n) ([Sa]) and an extension of the polynomial representation to DAHA-module. He proved irreducibility of the representation for generic parameters, and the duality relations. In [St], the evaluation formulas are shown by Stokman.

For rank $n = 1$, Oblomkov and Stoica classified finite dimensional representations of the DAHA of type (C_1^\vee, C_1) ([ObSt]) in the case when a parameter q is not a root of unity. They gave a description of subrepresentations in terms of the polynomial representation. The case when $q = 1$ and the other parameters are generic is studied in [Ob].

Recently, Cherednik gave a condition when the polynomial representation of the DAHA is irreducible ([Ch]) for reduced root systems and a generic

parameter q . He introduced a pairing on the representation, and determined irreducibility by the condition if the radical of the pairing is zero or not. He also determined whether the radical is zero or not by so-called affine exponents.

We will recall the proof of irreducibility of the polynomial representation for generic parameters. q -Dunkl-Cherednik operators \widehat{Y}_i are simultaneously diagonalizable on the representation. This property is called Y -semisimplicity. Y -eigenfunctions are called non-symmetric Macdonald polynomials. (For type (C_n^\vee, C_n) , they are also called non-symmetric Koornwinder polynomials.) There exist operators which send a Y -eigenfunction to other Y -eigenfunctions. These operators are called intertwiners. By applying the intertwiners, we see that any Y -eigenfunction is cyclic.

In this paper, we treat the DAHA of type (C_n^\vee, C_n) with $n \geq 2$. The algebra has 6 parameters q, t, a^*, b^*, c^* , and d^* . We study the polynomial representation for specialized parameters where the representation can be non- Y -semisimple or reducible. (The parameter q may be a root of unity.) For such specialized parameters, non-symmetric Koornwinder polynomials may have some poles, or the argument in terms of the intertwiners above is not enough. In order to overcome these difficulties, we introduce certain linear combinations of non-symmetric Koornwinder polynomials where the poles are cancelled out (we call them *modified polynomials*). Modified polynomials are generalized Y -eigenfunctions. We also introduce *modified intertwiners*. Using them, we give an inductive and combinatorial method to compute modified polynomials. So that, a linear basis of the polynomial representation is given in terms of the modified polynomials. The irreducibility is shown by checking that any non-zero vector in the representation is cyclic. (The notion “modified polynomials” in this paper are, in some sense, similar to *non-semisimple Macdonald polynomials* introduced in [Ch] for reduced root systems. The feature of the present paper is, as we stated above, to obtain an inductive and combinatorial method by the use of the modified intertwiners. See Remark 3.13 (ii).)

When the representation is reducible, we describe subrepresentations in terms of spanning sets of non-symmetric Koornwinder polynomials, or we realize them as ideals which are invariant under the action of DAHA.

The main statements in this paper are as follows:

Theorem 1.1. *(i) The polynomial representation is irreducible and Y -semisimple if the parameters q, t, a^*, b^*, c^*, d^* are not roots of the following Laurent polynomials:*

$$(1) \quad t^{(k+1)/m} q^{(r-1)/m} - \omega_m$$

$$(n \geq k+1 \geq 0, r-1 \geq 1, m = \text{GCD}(k+1, r-1),$$

$$\omega_m \text{ is a primitive } m\text{-th root of unity}),$$

$$\begin{aligned}
 (2) \quad & t^{k+1} q^{r-1} a^{*2} - 1 \\
 & (2n - 2 \geq k + 1 \geq 0, r - 1 \geq 1), \\
 (3) \quad & t^{n-i} q^{r-1} a^* b^{*\pm 1} - 1 \\
 & (n \geq i \geq 1, r - 1 \geq 1), \\
 (4) \quad & t^{n-i} q^{r-1-\theta(\pm 1)} a^* c^{*\pm 1} - 1 \\
 & (n \geq i \geq 1, r - 1 \geq 1), \\
 (5) \quad & t^{n-i} q^{r-1-\theta(\pm 1)} a^* d^{*\pm 1} - 1 \\
 & (n \geq i \geq 1, r - 1 \geq 1)
 \end{aligned}$$

where $\theta(+1) = 1$ and $\theta(-1) = 0$.

(ii) We take a specialization of parameters such that the parameters are roots of one of (1), (2), (3), (4), or (5) and are not roots of the others. Under the specialization, we give a linear basis of the polynomial representation in terms of linear combinations of non-symmetric Koornwinder polynomials. (The linear combinations are of the form (21). The construction of the basis is given in Theorem 3.14.)

(iii-1) For the case (1) with $k + 1 \geq 2$, the polynomial representation is reducible. It is Y -semisimple if $k + 1 = n$, and it is not Y -semisimple if $2 \leq k + 1 < n$. We give $N = \lfloor \frac{n}{k+1} \rfloor$ subrepresentations $I_1^{(k,r)} \subset I_2^{(k,r)} \subset \dots \subset I_N^{(k,r)}$ in terms of vanishing conditions: a Laurent polynomial $f(z_1, \dots, z_n)$ should be zero if certain ratios or products of z_i take certain values (see Definition 4.11). We show that $I_1^{(k,r)}$ is irreducible and Y -semisimple. A basis of $I_1^{(k,r)}$ is given (see Theorem 4.6 and Theorem 4.14). Detailed arguments are given in §4.2.

(iii-2) For the case (1) with $k + 1 = 1$ or (2), the polynomial representation is irreducible and not Y -semisimple. Detailed arguments are given in §4.5, §4.6.

(iii-3) For the case (3) or (4) or (5), the polynomial representation is reducible and Y -semisimple. It has a unique subrepresentation. If the sign is plus in (3), (4), (5), then the subrepresentation is characterized in terms of a vanishing condition: a Laurent polynomial $f(z_1, \dots, z_n)$ should be zero if (z_1, \dots, z_n) are included in certain grid points. This vanishing condition is a generalization of that in [vDSt]. The explicit statements and detailed arguments are given in §4.7, §4.8.

(iii-4) For the case (1) with $k + 1 = 0$ (that is, q is a root of unity), the polynomial representation is reducible and Y -semisimple. There are infinitely-many subrepresentations labelled by partitions with length $\leq n$, and inclusion relation of two subrepresentations is determined by the dominance ordering of corresponding partitions. All the irreducible subquotients are finite dimensional, and isomorphic to each other. Detailed arguments are given in §4.9.

As an application, we show some non-symmetric Koornwinder polynomials for the specialized parameters are eigenfunctions with respect to Demazure-Lusztig operators \hat{T}_i for some i . For such Laurent polynomials, one can construct polynomial solutions of quantum Knizhnik-Zamolodchikov (qKZ) equation of type (C_n^\vee, C_n) (about detail, see [KaSh]). In Proposition 5.4 of [KaTa], an example of polynomial solutions of qKZ equation (of type GL_n) is given. In this paper, in Proposition 4.16, we give a generalization of the example in [KaTa].

On the result (iii-1), there are some variants for symmetric (Laurent) polynomials or for the polynomial representation of GL_n -DAHA.

In [FJMM], the terminology “wheel condition” originally appeared. They considered a vanishing condition for \mathfrak{S}_n -symmetric polynomials and called it *wheel condition*. They give a linear basis of the ideal defined by the wheel condition in terms of symmetric Macdonald polynomials for specialized parameters.

In [Ka1], standing on [FJMM], the author introduced a vanishing condition for BC_n -symmetric Laurent polynomials, and give a linear basis of the ideal in terms of symmetric Macdonald-Koornwinder polynomials for specialized parameters. For BC_n -symmetric Laurent polynomials, the vanishing condition in [Ka1] is equivalent to 1-wheel condition in the present paper.

For the DAHA of type GL_n , the algebra has only two parameters q and t . When the parameters are specialized at $t^{(k+1)/m}q^{(r-1)/m} = \omega_m$ (where $n \geq k+1 \geq 2$, $r-1 \geq 1$, $m = \text{GCD}(k+1, r-1)$, ω_m is a primitive m -th root of unity), the author introduced vanishing conditions for Laurent polynomials called multi-wheel condition (of type GL_n) in [Ka2]. By this condition, a series of subrepresentations of the polynomial representation was constructed. In [En], it was shown that the series gives the composition series of the polynomial representation for the case $m = \text{GCD}(k+1, r-1) = 1$ and $k+1 \neq 2$. The result (iii-1) in the present paper is a (C_n^\vee, C_n) -version of the result in [Ka2]. (The cases (2), ..., (5) have no counterparts for GL_n -DAHA due to the difference of numbers of parameters.)

This paper is organized as follows.

In Section 2, we introduce notations in this paper and review basic properties for DAHA and its polynomial representation. Especially, duality of the algebra, the Noumi representation, the non-symmetric Koornwinder polynomials, the intertwiners, irreducibility of the representation, duality relations and evaluation formulas for the non-symmetric Koornwinder polynomials are stated.

In Section 3, we consider specializations of parameters where the representation can be non- Y -semisimple or reducible. We realize such specializations as follows: First, take an irreducible factor s of (1), (2), (3), (4), or (5). Second, take the quotient of the Laurent polynomial ring of the parameters by

the prime ideal generated by s . Finally, take the fractional field of the quotient ring. This realization gives a field where the parameters satisfy $s = 0$. For such a specialization, we introduce modified intertwiners, and in terms of them, we give a method to compute modified polynomials (generalized Y -eigenfunctions). As a result, a linear basis of the polynomial representation is given by the modified polynomials.

In Section 4, applying tools given in Section 3, we examine irreducibility and Y -semisimplicity of the polynomial representation for each case of (1), (2), (3), (4), or (5). For the cases where the representation is reducible, we describe subrepresentations as spanning sets of the non-symmetric Koornwinder polynomials or as ideals defined by certain vanishing conditions. The duality relations and the evaluation formulas give values of the non-symmetric Koornwinder polynomial at certain grid points. By using this property, we show that some non-symmetric Koornwinder polynomials satisfy the vanishing conditions.

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2. NOTATIONS AND BASIC PROPERTIES

2.1. The double affine Hecke algebra. In this paper, we assume $n \geq 2$. We realize the affine root system of type C_n in $\mathbb{R}^n \oplus \mathbb{R}\delta$ and denote affine simple roots by:

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq n-1), \quad \alpha_n = 2\epsilon_n, \quad \alpha_0 = \delta - 2\epsilon_1,$$

where ϵ_i are standard bases: $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ and δ is a radical element. Let $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ be simple co-roots and let $\varpi_i = \epsilon_1 + \dots + \epsilon_i$ be the fundamental weights and $\Lambda_n \cong \mathbb{Z}^n$ be the weight lattice.

Let $\mathbb{K} = \mathbb{C}(q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}, u_n^{1/2}, u_0^{1/2})$. Denote the ring of n -variable Laurent polynomials by $P_n = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. P_n is isomorphic to the group algebra $\mathbb{K}\Lambda_n$. Denote the finite and affine Weyl groups of type C_n by $W_0 = \langle s_1, \dots, s_n \rangle$, $W = \langle s_0, \dots, s_n \rangle$, respectively. The action of the affine Weyl group is given by $wx^\lambda = x^{w\lambda}$ where $x^\delta = q$. Especially,

$$\begin{aligned} s_0 f(x_1, x_2, \dots) &= f(qx_1^{-1}, x_2, \dots) \\ s_n f(\dots, x_{n-1}, x_n) &= f(\dots, x_{n-1}, x_n^{-1}). \end{aligned}$$

Definition 2.1 (DAHA). The double affine Hecke algebra (DAHA) $\mathcal{H}_n = \mathcal{H}_n(q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}, u_n^{1/2}, u_0^{1/2})$ of type (C_n^\vee, C_n) ([Sa]) is a unital associative \mathbb{K} -algebra generated by $T_0, \dots, T_n, X_1^{\pm 1}, \dots, X_n^{\pm 1}$ with defining relations

as follows:

quadratic Hecke relations:

$$\begin{aligned} (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) &= 0, \\ (T_i - t_i^{1/2})(T_i + t_i^{-1/2}) &= 0 \quad 1 \leq i \leq n-1, \\ (T_n - t_n^{1/2})(T_n + t_n^{-1/2}) &= 0, \end{aligned}$$

braid relations:

$$\begin{aligned} T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad 1 \leq i \leq n-2, \\ T_{n-1} T_n T_{n-1} T_n &= T_n T_{n-1} T_n T_{n-1}, \\ T_i T_j &= T_j T_i \quad |i-j| \geq 2, \end{aligned}$$

relations between X and T :

$$\begin{aligned} X_i X_j &= X_j X_i \quad \forall i, j, \\ T_i X_j &= X_j T_i \quad \langle \alpha_i, \epsilon_j \rangle = 0, \\ T_i X_i T_i &= X_{i+1} \quad 1 \leq i \leq n-1, \\ X_n^{-1} T_n^{-1} &= T_n X_n + (u_n^{1/2} - u_n^{-1/2}), \\ q^{-1/2} T_0^{-1} X_1 &= q^{1/2} X_1^{-1} T_0 + (u_0^{1/2} - u_0^{-1/2}). \end{aligned}$$

The subalgebra $\mathcal{H}_n^{\text{aff}} = \mathcal{H}_n^{\text{aff}}(t^{1/2}, t_n^{1/2} t_0^{1/2}) \subset \mathcal{H}_n$ generated by T_0, \dots, T_n is the affine Hecke algebra of type C_n . Let $\mathcal{H}_n^Y = \mathcal{H}_n^Y(t^{1/2}, t_n^{1/2} t_0^{1/2})$ be the algebra generated by T_1, \dots, T_n and $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$ with the defining relations as follows:

$$\begin{aligned} \text{the quadratic Hecke relations} &\text{ for } T_1, \dots, T_n, \\ \text{the braid relations} &\text{ for } T_1, \dots, T_n, \\ Y_i Y_j &= Y_j Y_i \quad \forall i, j, \\ T_i Y_j &= Y_j T_i \quad \langle \alpha_i, \epsilon_j \rangle = 0, \\ T_i Y_{i+1} T_i &= Y_i \quad 1 \leq i \leq n-1, \\ T_n^{-1} Y_n &= Y_n^{-1} T_n + (t_0^{1/2} - t_0^{-1/2}). \end{aligned}$$

Then \mathcal{H}_n^Y is isomorphic to $\mathcal{H}_n^{\text{aff}}$ by the correspondence

$$Y_i \mapsto T_i \dots T_{n-1} T_n \dots T_0 T_1^{-1} \dots T_{i-1}^{-1}.$$

Hereafter we identify Y_i with the right hand side above. We see that the elements T_i and Y_j satisfy the following relation:

$$\begin{aligned} q^{-1} Y_1^{-1} U_n^{-1} &= U_n Y_1 + q^{-1/2} (u_0^{1/2} - u_0^{-1/2}) \\ \text{where} \\ U_n &:= X_1^{-1} T_0 Y_1^{-1}. \end{aligned}$$

Let $*$: $\mathbb{K} \rightarrow \mathbb{K}$ be an involution given by

$$t_0^{1/2} \leftrightarrow u_n^{1/2}$$

and the other parameters $t^{1/2}, q^{1/2}, t_n^{1/2}, u_0^{1/2}$ are fixed. There is an anti-automorphism of DAHA:

$$* : \mathcal{H}_n(q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}, u_n^{1/2}, u_0^{1/2}) \rightarrow \mathcal{H}_n(q^{1/2}, t^{1/2}, t_n^{1/2}, u_n^{1/2}, t_0^{1/2}, u_0^{1/2})$$

given by

$$T_0 \mapsto U_n, T_i \mapsto T_i, X_i \mapsto Y_i^{-1}, Y_i \mapsto X_i^{-1} \quad (i = 1, \dots, n).$$

The anti-automorphism $*$ is called duality anti-involution. Especially, the subalgebra $\mathcal{H}_n^X \subset \mathcal{H}_n$ generated by T_1, \dots, T_n and X_1, \dots, X_n is anti-isomorphic to the algebra $\mathcal{H}_n^Y(t^{1/2}, t_n^{1/2} u_n^{1/2})$. The parameters

$$(q^{1/2*}, t^{1/2*}, t_n^{1/2*}, t_0^{1/2*}, u_n^{1/2*}, u_0^{1/2*}) = (q^{1/2}, t^{1/2}, t_n^{1/2}, u_n^{1/2}, t_0^{1/2}, u_0^{1/2})$$

are called dual parameters.

We often use another notations for parameters ([Sa]):

$$a = t_n^{1/2} u_n^{1/2}, b = -t_n^{1/2} u_n^{-1/2}, c = q^{1/2} t_0^{1/2} u_0^{1/2}, d = -q^{1/2} t_0^{1/2} u_0^{-1/2}.$$

$$a' = t_n^{-1/2} t_0^{-1/2}, b' = -t_n^{-1/2} t_0^{1/2}, c' = q^{-1/2} u_n^{-1/2} u_0^{-1/2}, d' = -q^{-1/2} u_n^{-1/2} u_0^{1/2}.$$

$$a^* = t_n^{1/2} t_0^{1/2}, b^* = -t_n^{1/2} t_0^{-1/2}, c^* = q^{1/2} u_n^{1/2} u_0^{1/2}, d^* = -q^{1/2} u_n^{1/2} u_0^{-1/2}.$$

Note that a^*, \dots, d^* are dual parameters of a, \dots, d , and a', \dots, d' are inverses of a^*, \dots, d^* .

An action of \mathcal{H}_n is realized on the ring of n -variable Laurent polynomials P_n .

Proposition 2.2 (polynomial representation). *Define the linear operators on P_n ([No]):*

$$\begin{aligned} \widehat{T}_0^{\pm 1} &= t_0^{\pm 1/2} + t_0^{-1/2} \frac{(1 - cx_1^{-1})(1 - dx_1^{-1})}{1 - qx_1^{-2}} (s_0 - 1) \\ \widehat{T}_i^{\pm 1} &= t^{\pm 1/2} + t^{-1/2} \frac{1 - tx_i x_{i+1}^{-1}}{1 - x_i x_{i+1}^{-1}} (s_i - 1) \\ \widehat{T}_n^{\pm 1} &= t_n^{\pm 1/2} + t_n^{-1/2} \frac{(1 - ax_n)(1 - bx_n)}{1 - x_n^2} (s_n - 1). \end{aligned}$$

The map $T_i \mapsto \widehat{T}_i$ and $X_j \mapsto x_j$ ($0 \leq i \leq n, 1 \leq j \leq n$) gives a representation of DAHA ([Sa]). This is called the polynomial representation, or the Noumi representation.

2.2. The non-symmetric Koornwinder polynomials. It is known that the q-Dunkl-Cherednik operators \widehat{Y}_i given by

$$\widehat{Y}_i := \widehat{T}_i \dots \widehat{T}_{n-1} \widehat{T}_n \dots \widehat{T}_0 \widehat{T}_1^{-1} \dots \widehat{T}_{i-1}^{-1}$$

are mutually commutative and triangular with respect to an partial ordering and the pairs of eigenvalues of $\widehat{Y}_1, \dots, \widehat{Y}_n$ are mutually different. Consequently, the space P_n is simultaneously diagonalizable with respect to $\widehat{Y}_1, \dots, \widehat{Y}_n$. Their joint eigenfunctions are called non-symmetric Koornwinder polynomials. We call this property Y -semisimplicity.

We will explain the explicit definition of the ordering, and the eigenvalues, and so on.

Denote by λ^+ the unique dominant element in $W_0\lambda$. (λ^+ is a partition of length $\leq n$.) Take the shortest element $w \in W_0$ such that $w\lambda^+ = \lambda$ and denote it by w_λ^+ . Put $\rho = (n-1, n-2, \dots, 1, 0)$, $\rho(\lambda) = w_\lambda^+ \rho$, $\sigma(\lambda) = (\text{sgn}(\lambda_1), \dots, \text{sgn}(\lambda_n))$ where $\text{sgn}(0) = +1$. We identify any element $w \in W_0$ with a permutation $\{\pm 1, \dots, \pm n\} \rightarrow \{\pm 1, \dots, \pm n\}$ by $\pm i \mapsto \langle (\sum_{j=1}^n j\epsilon_j), \pm w\epsilon_i \rangle$. Then we see that $\text{sgn}(w_\lambda^+(i)) = \text{sgn}(\lambda_i)$. We extend the action of W_0 on \mathbb{R}^n to an action of W on \mathbb{R}^n by

$$s_0 \cdot (v_1, v_2, \dots, v_n) = (-1 - v_1, v_2, \dots, v_n).$$

We call it the *dot* action of W .

Define partial orderings $\lambda \geq \mu$ and $\lambda \succeq \mu$ in \mathbb{Z}^n as follows:

$$\begin{aligned} \lambda \geq \mu & \quad \text{if } \lambda - \mu \in \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i^\vee, \\ \lambda \succeq \mu & \quad \text{if } \lambda^+ > \mu^+, \text{ or } \lambda^+ = \mu^+ \text{ and } \lambda \geq \mu. \end{aligned}$$

Definition 2.3 (non-symmetric Koornwinder polynomials ([No, Sa, St])). For $\lambda \in \mathbb{Z}^n$, the non-symmetric Koornwinder polynomial E_λ is defined by

$$\begin{aligned} \widehat{Y}_i E_\lambda &= y(\lambda)_i E_\lambda \\ E_\lambda &= x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda\mu} x^\mu \quad (c_{\lambda\mu} \in \mathbb{K}) \end{aligned}$$

where

$$y(\lambda)_i := q^{\lambda_i} t^{\rho(\lambda)_i} a^{*\sigma(\lambda)_i}.$$

For the pair of eigenvalues $y(\lambda) = (y(\lambda)_1, \dots, y(\lambda)_n)$, and any Laurent polynomial (or rational function) f , we express the scalar $f(y(\lambda)) \in \mathbb{K}$ by $f(Y)|_\lambda$ or $f(\lambda)$:

$$\begin{aligned} f(Y)|_\lambda &\stackrel{\text{def}}{=} f(y(\lambda)) \\ f(\lambda) &\stackrel{\text{def}}{=} f(y(\lambda)). \end{aligned}$$

We denote by E_λ^* the polynomial where the parameters of E_λ are replaced by the dual parameters. We see that $\{E_\lambda; \lambda \in \mathbb{Z}^n\}$ or $\{E_\lambda^*; \lambda \in \mathbb{Z}^n\}$ constitutes a \mathbb{K} -basis of P_n .

2.3. The intertwiners. There are operators which send an Y -eigenfunction to other Y -eigenfunctions. Such operators are called intertwining operators or intertwiners.

Definition 2.4 (intertwiners). *Let ϕ_i be as follows:*

$$\begin{aligned}\phi_i &= T_i + \frac{t^{1/2} - t^{-1/2}}{Y_{i+1}/Y_i - 1} \quad (1 \leq i \leq n-1), \\ \phi_n &= T_n + \frac{(t_n^{1/2} - t_n^{-1/2}) + (t_0^{1/2} - t_0^{-1/2})Y_n^{-1}}{Y_n^{-2} - 1}, \\ \phi_0 &= U_n + \frac{(u_n^{1/2} - u_n^{-1/2}) + (u_0^{1/2} - u_0^{-1/2})q^{1/2}Y_1}{qY_1^2 - 1}.\end{aligned}$$

Since there are rational functions $f(Y)$ in the right hand sides, these operators ϕ_i are not elements of \mathcal{H}_n . However ϕ_i are considered as linear operators in P_n by replacing $f(Y)$ by $f(Y)|_\lambda$ on each eigenspace $\mathbb{K}E_\lambda$.

For any $\lambda + m\delta \in \mathbb{Z}^n \oplus \frac{1}{2}\mathbb{Z}\delta$, we define $Y^{\lambda+m\delta} = q^{-m}Y_1^{\lambda_1} \cdots Y_n^{\lambda_n}$. Define 1-variable Laurent polynomials D_i and N_i ($1 \leq i \leq n$) by $D_i(x) = x^{-1} - 1$ ($0 \leq i \leq n$) and

$$\begin{aligned}N_i(x) &= t^{1/2}(x^{-1} - t^{-1}) \quad (1 \leq i \leq n-1), \\ N_n(x) &= t_n^{1/2}(x^{-1} - a')(x^{-1} - b'), \\ N_0(x) &= u_n^{1/2}(x^{-1} - q^{1/2}c')(x^{-1} - q^{1/2}d').\end{aligned}$$

Then $D_i(Y^{\pm\alpha_i^\vee})$ and $N_i(Y^{\pm\alpha_i^\vee})$ are given as follows:

$$\begin{aligned}D_i(Y^{\pm\alpha_i^\vee}) &= Y_{i+1}^{\pm 1}Y_i^{\mp 1} - 1 \quad (1 \leq i \leq n-1) \\ (6) \quad D_n(Y^{\pm\alpha_n^\vee}) &= Y_n^{\mp 2} - 1 \\ D_0(Y^{\pm\alpha_0^\vee}) &= q^{\pm 1}Y_1^{\pm 2} - 1\end{aligned}$$

and

$$\begin{aligned}N_i(Y^{\pm\alpha_i^\vee}) &= t^{1/2}(Y_{i+1}^{\pm 1}Y_i^{\mp 1} - t^{-1}) \quad (1 \leq i \leq n-1) \\ (7) \quad N_n(Y^{\pm\alpha_n^\vee}) &= t_n^{1/2}(Y_n^{\mp 1} - a')(Y_n^{\mp 1} - b') \\ N_0(Y^{\pm\alpha_0^\vee}) &= u_n^{1/2}(q^{\pm 1/2}Y_1^{\pm 1} - q^{1/2}c')(q^{\pm 1/2}Y_1^{\pm 1} - q^{1/2}d').\end{aligned}$$

Note that $D_i(Y^{\alpha_i^\vee})$ is the denominator of ϕ_i .

Proposition 2.5. *For any $0 \leq i \leq n$, we have*

$$\begin{aligned}\phi_i Y^{\epsilon_j} &= Y^{s_i \epsilon_j} \phi_i \quad (1 \leq j \leq n), \\ \phi_i^2 &= \frac{N_i(Y^{\alpha_i^\vee})N_i(Y^{-\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})D_i(Y^{-\alpha_i^\vee})}, \\ \phi_i \phi_{i+1} \phi_i &= \phi_{i+1} \phi_i \phi_{i+1} \quad (1 \leq i \leq n-2), \\ \phi_i \phi_{i+1} \phi_i \phi_{i+1} &= \phi_{i+1} \phi_i \phi_{i+1} \phi_i \quad (i = 0, n-1).\end{aligned}$$

We see that $\phi_i E_\lambda$ is proportional to $E_{s_i \cdot \lambda}$. Let $c_{i,\lambda}$ be the coefficient

$$\phi_i E_\lambda = c_{i,\lambda} E_{s_i \cdot \lambda}.$$

Then for $1 \leq i \leq n-1$,

$$\begin{aligned} c_{i,\lambda} &= t^{1/2} && \text{if } \langle \lambda, \alpha_i \rangle < 0, \\ c_{i,\lambda} &= 0 && \text{if } \langle \lambda, \alpha_i \rangle = 0, \\ c_{i,\lambda} &= t^{-1/2} \phi_i^2|_\lambda && \text{if } \langle \lambda, \alpha_i \rangle > 0, \end{aligned}$$

and

$$\begin{aligned} c_{n,\lambda} &= t_n^{1/2} && \text{if } \langle \lambda, \alpha_n \rangle < 0, \\ c_{n,\lambda} &= 0 && \text{if } \langle \lambda, \alpha_n \rangle = 0, \\ c_{n,\lambda} &= t_n^{-1/2} \phi_n^2|_\lambda && \text{if } \langle \lambda, \alpha_n \rangle > 0, \\ c_{0,\lambda} &= t_0^{1/2} t^{-\rho(\lambda)_1} a^{*-1} && \text{if } \langle \lambda, \alpha_0 \rangle \leq 0, \text{ that is } \lambda_1 \geq 0, \\ c_{0,\lambda} &= t_0^{-1/2} t^{\rho(\lambda)_1} a^* \phi_0^2|_\lambda && \text{if } \langle \lambda, \alpha_0 \rangle > 0, \text{ that is } \lambda_1 < 0. \end{aligned}$$

Note that $c_{i,\lambda}$ are Laurent monomials in \mathbb{K} if $\langle \lambda, \alpha_i \rangle \leq 0$ and $c_{i,\lambda}$ are rational functions in \mathbb{K} if $\langle \lambda, \alpha_i \rangle > 0$.

Remark 2.6. Originally, Sahi introduced the intertwining operators $S_i \in \mathcal{H}_n$ for type (C_n^\vee, C_n) ([Sa]) by

$$S_i = [T_i, Y_i], \quad S_n = [T_n, Y_n], \quad S_0 = [Y_1, U_n].$$

The relations between S_i and ϕ_i are given by $S_i = \phi_i \cdot (Y_i - Y_{i+1})$, $S_n = \phi_n \cdot (Y_n - Y_n^{-1})$, $S_0 = (Y_1 - q^{-1} Y_1^{-1}) \cdot \phi_0$. In [St] or [NUKW], similar intertwiners S_i are also introduced. However, the coefficient $c'_{i,\lambda}$ in $S_i E_\lambda = c'_{i,\lambda} E_{s_i \cdot \lambda}$ becomes more complicated even if $\langle \lambda, \alpha_i \rangle < 0$. We introduce ϕ_i in order to take the coefficient $c_{i,\lambda}$ as a Laurent monomial in \mathbb{K} if $\langle \lambda, \alpha_i \rangle < 0$.

Since the dot action of W on \mathbb{Z}^n is transitive, we can check that any E_λ ($\lambda \in \mathbb{Z}^n$) is a cyclic vector in P_n by applying the intertwiners ϕ_i . Hence, we obtain the statement as follows:

Proposition 2.7 (irreducibility([Sa])). *The polynomial representation P_n is irreducible.*

2.4. The duality relations and the evaluation formulas. We introduce two properties called duality relations and evaluation formulas for E_λ .

Definition 2.8. For $\lambda \in \mathbb{Z}^n$ and $f \in P_n$, we define two maps χ_λ and χ_λ^* : $P_n \rightarrow \mathbb{K}$ as follows:

$$\begin{aligned} \chi_\lambda(f) &= f(y(\lambda)_1^{-1}, \dots, y(\lambda)_n^{-1}) \\ \chi_\lambda^*(f) &= f(y^*(\lambda)_1^{-1}, \dots, y^*(\lambda)_n^{-1}). \end{aligned}$$

Proposition 2.9 (duality relations([Sa])). *For any $\lambda, \mu \in \mathbb{Z}^n$, we have*

$$\chi_\mu^*(E_\lambda) \chi_0(E_\mu^*) = \chi_\lambda(E_\mu^*) \chi_0^*(E_\lambda).$$

We have relations between $\chi_0^*(E_{s_i \cdot \lambda})$ and $\chi_0^*(E_\lambda)$.

Lemma 2.10 (recurrence relations). *(Recall (6) and (7) for the definition of D_i and N_i .) If $\langle \lambda, \alpha_i \rangle < 0$ ($1 \leq i \leq n-1$), then*

$$\chi_0^*(E_{s_i \cdot \lambda}) = t^{-1/2} \frac{N_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})} \Big|_\lambda \chi_0^*(E_\lambda).$$

If $\langle \lambda, \alpha_n \rangle < 0$, then

$$\chi_0^*(E_{s_n \cdot \lambda}) = t_n^{-1/2} \frac{N_n(Y^{\alpha_n^\vee})}{D_n(Y^{\alpha_n^\vee})} \Big|_\lambda \chi_0^*(E_\lambda).$$

If $\langle \lambda, \alpha_0 \rangle < 0$, (that is $\lambda_1 \geq 0$), then

$$\chi_0^*(E_{s_0 \cdot \lambda}) = t_0^{-1/2} t^{\rho(\lambda)_1} a^* \frac{N_0(Y^{\alpha_0^\vee})}{D_0(Y^{\alpha_0^\vee})} \Big|_\lambda \chi_0^*(E_\lambda).$$

Proof. The statements are shown by computing $\chi_0^*(\phi_i E_\lambda)$ ($0 \leq i \leq n$). \square

Proposition 2.11 (evaluation formula([St])). *Suppose $\lambda = \lambda^+$. Then we have*

$$\begin{aligned} \chi_0^*(E_\lambda) &= \prod_{i < j} t^{-(\lambda_i - \lambda_j)} \frac{(t^{j-i+1} q; q)_{\lambda_i - \lambda_j}}{(t^{j-i} q; q)_{\lambda_i - \lambda_j}} \\ &\times \prod_{i < j} t^{-(\lambda_i + \lambda_j)} \frac{(t^{2n-i-j+1} a^{*2} q; q)_{\lambda_i + \lambda_j}}{(t^{2n-i-j} a^{*2} q; q)_{\lambda_i + \lambda_j}} \\ &\times \prod_i a^{-\lambda_i} t^{(n-i)\lambda_i} \frac{(t^{n-i} a^{*2} q, t^{n-i} a^* b^* q, t^{n-i} a^* c^*, t^{n-i} a^* d^*; q)_{\lambda_i}}{(q^2 t^{2(n-i)} a^{*2}, q t^{2(n-i)} a^{*2}; q^2)_{\lambda_i}}, \end{aligned}$$

where $(x; q)_i = \prod_{l=0}^{i-1} (1 - xq^l)$ and $(x_1, \dots, x_j; q)_i = \prod_{l=1}^j (x_l; q)_i$.

Proof. This formula is true for $\lambda = (0, \dots, 0)$. We can show that it is true for any $\lambda = \lambda^+$ inductively, using the recurrence relations for $\chi_0^*(E_\lambda)$. \square

3. SPECIALIZATION OF PARAMETERS

In this section, we introduce a specialization of parameters where the polynomial representation can be non- Y -semisimple or reducible. Put $\mathcal{A} = \mathbb{C}[q^{\pm 1/2}, t^{\pm 1/2}, t_n^{\pm 1/2}, t_0^{\pm 1/2}, u_n^{\pm 1/2}, u_0^{\pm 1/2}]$. (\mathbb{K} is the fractional field of \mathcal{A} .)

3.1. Generic and specialized parameters. We consider a specialization of parameters by replacing the field \mathbb{K} by some field including \mathbb{C} . Throughout this paper, we always assume that $q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}, u_n^{1/2}, u_0^{1/2} \neq 0$ under any specialization of the parameters.

Proposition 3.1 (generic parameters). *The polynomial representation for specialized parameters is Y -semisimple and irreducible if the parameters are not roots of either of Laurent polynomials given as follows:*

$$\begin{aligned}
(8) \quad & t^{(k+1)/m} q^{(r-1)/m} - \omega_m \\
& (n \geq k+1 \geq 0, r-1 \geq 1, m = \text{GCD}(k+1, r-1), \\
& \omega_m \text{ is a primitive } m\text{-th root of unity}), \\
(9) \quad & t^{k+1} q^{r-1} a^{*2} - 1 \\
& (2n-2 \geq k+1 \geq 0, r-1 \geq 1), \\
(10) \quad & t^{n-i} q^{r-1} a^* b^{*\pm 1} - 1 \\
& (n \geq i \geq 1, r-1 \geq 1), \\
(11) \quad & t^{n-i} q^{r-1-\theta(\pm 1)} a^* c^{*\pm 1} - 1 \\
& (n \geq i \geq 1, r-1 \geq 1), \\
(12) \quad & t^{n-i} q^{r-1-\theta(\pm 1)} a^* d^{*\pm 1} - 1 \\
& (n \geq i \geq 1, r-1 \geq 1).
\end{aligned}$$

Note that $a^{*2} = t_n t_0$, $a^* b^* = -t_n$, $a^* b^{*-1} = -t_0$, and

$$\begin{aligned}
q^{-\theta(\pm 1)} a^* c^{*\pm 1} &= q^{-1/2} t_n^{1/2} t_0^{1/2} u_n^{\pm 1/2} u_0^{\pm 1/2}, \\
q^{-\theta(\pm 1)} a^* d^{*\pm 1} &= -q^{-1/2} t_n^{1/2} t_0^{1/2} u_n^{\pm 1/2} u_0^{\mp 1/2}.
\end{aligned}$$

In order to prove Proposition 3.1, we give a sufficient condition where all Y -eigenvalues do not degenerate.

Lemma 3.2. *Suppose that $y(\lambda) = y(\mu)$ under a specialization of parameters for some $\lambda \neq \mu \in \mathbb{Z}^n$. Then the parameters are roots of either of the following Laurent polynomials:*

$$\begin{aligned}
(13) \quad & t^{k+1} q^{r-1} - 1 \quad (n-1 \geq k+1 \geq 0, r-1 \geq 1), \\
(14) \quad & t^{k+1} q^{r-1} a^{*2} - 1 \quad (2n-2 \geq k+1 \geq 0, r-1 \geq 1).
\end{aligned}$$

Proof. We introduce a notation $\mathbf{y}(\lambda)_i := (\lambda_i, \rho(\lambda)_i, \sigma(\lambda)_i)$ for any $\lambda \in \mathbb{Z}^n$. Suppose that $y(\lambda) = y(\mu)$ under a specialization of parameters for some $\lambda \neq \mu$. Let i_1, \dots, i_n be

$$(i_1, \dots, i_n) = (|w_\lambda^+(1)|, \dots, |w_\lambda^+(n)|).$$

Then, $|\rho(\lambda)_{i_m}| = n - i_m$ for any $1 \leq m \leq n$. Fix $1 \leq \ell \leq n$ such that $\mathbf{y}(\lambda)_{i_\ell} \neq \mathbf{y}(\mu)_{i_\ell}$ and $\mathbf{y}(\lambda)_{i_m} = \mathbf{y}(\mu)_{i_m}$ for any $1 \leq m < \ell$.

We divide the proof by the signs of $(\sigma(\lambda)_{i_\ell}, \sigma(\mu)_{i_\ell})$.

(i) If $(\sigma(\lambda)_{i_\ell}, \sigma(\mu)_{i_\ell}) = (+, -)$, then the condition $y(\lambda)_{i_\ell} = y(\mu)_{i_\ell}$ implies that

$$q^{\lambda_{i_\ell} - \mu_{i_\ell}} t^{\rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell}} a^{*2} = 1.$$

Since $(\lambda_{i_\ell} - \mu_{i_\ell}) \geq 0 - (-1) = 1$ and $2n-2 \geq \rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell} \geq 0$, the parameters are roots of (14).

(ii) The proof for the case $(\sigma(\lambda)_{i_\ell}, \sigma(\mu)_{i_\ell}) = (-, +)$ is similar to that for the case $(+, -)$.

(iii) If $(\sigma(\lambda)_{i_\ell}, \sigma(\mu)_{i_\ell}) = (+, +)$, then the condition $y(\lambda)_{i_\ell} = y(\mu)_{i_\ell}$ implies that

$$(15) \quad q^{\lambda_{i_\ell} - \mu_{i_\ell}} t^{\rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell}} = 1.$$

By the definition of i_ℓ , we see $n - 1 \geq \rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell} \geq 0$. If $\rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell} = 0$, then since $\mathbf{y}(\lambda)_{i_\ell} \neq \mathbf{y}(\mu)_{i_\ell}$, we see that $\lambda_{i_\ell} \neq \mu_{i_\ell}$. Thus (15) implies that the parameters are roots of (13). Assume that $n - 1 \geq \rho(\lambda)_{i_\ell} - \rho(\mu)_{i_\ell} \geq 1$. If $\lambda_{i_\ell} - \mu_{i_\ell} \geq 1$, then (15) implies that the parameters are roots of (13).

Hence assume that $\lambda_{i_\ell} \leq \mu_{i_\ell}$. Take the index j such that $|\rho(\mu)_j| = n - i_\ell$. Since $|\rho(\mu)_j| = n - i_\ell = \rho(\lambda)_{i_\ell} > \rho(\mu)_{i_\ell}$, we see that $|\mu_j| \geq \mu_{i_\ell}$ and $j \neq i_\ell$. Since $|\rho(\lambda)_j| < n - i_\ell = \rho(\lambda)_{i_\ell}$, we have $|\lambda_j| \leq \lambda_{i_\ell}$. Therefore, $|\lambda_j| \leq \lambda_{i_\ell} \leq \mu_{i_\ell} \leq |\mu_j|$. If the sign $\sigma(\lambda)_j$ is opposite to the sign $\sigma(\mu)_j$, then by the same argument above, the condition $y(\lambda)_j = y(\mu)_j$ implies that the parameters are roots of (14). If $(\sigma(\lambda)_j, \sigma(\mu)_j) = (+, +)$, then the condition $y(\lambda)_j = y(\mu)_j$ implies that

$$q^{\mu_j - \lambda_j} t^{\rho(\mu)_j - \rho(\lambda)_j} = 1$$

where $\mu_j - \lambda_j > 0$ and $n - 1 \geq \rho(\mu)_j - \rho(\lambda)_j > 0$. Therefore the parameters are roots of (13). If $(\sigma(\lambda)_j, \sigma(\mu)_j) = (-, -)$, then the condition $y(\lambda)_j = y(\mu)_j$ implies that

$$q^{\lambda_j - \mu_j} t^{\rho(\lambda)_j - \rho(\mu)_j} = 1$$

where $\lambda_j - \mu_j > 0$ and $n - 1 \geq \rho(\lambda)_j - \rho(\mu)_j > 0$. Therefore the parameters are roots of (13).

(iv) The proof for the case $(\sigma(\lambda)_{i_\ell}, \sigma(\mu)_{i_\ell}) = (-, -)$ is similar to that for the case $(+, +)$. \square

We will start the proof of Proposition 3.1. For parameters satisfying the assumption, the Y -semisimplicity and irreducibility of P_n are shown by the following statements: for any $\lambda \in \mathbb{Z}^n$,

- $y(\lambda) \neq y(\mu)$ for any $\mu \in \mathbb{Z}^n$ such that $\mu \neq \lambda$ and E_λ is well-defined under the specialization of parameters.
- $E_{s_i \cdot \lambda} \in \mathcal{H}_n E_\lambda$.
- The dot-action \cdot of W on \mathbb{Z}^n is transitive.

The first statement is already shown by Lemma 3.2, and the third statement is easy.

Proof of Proposition 3.1. (See (6) and (7) for the definition of $D_i(Y^{\pm \alpha_i^\vee})$ and $N_i(Y^{\pm \alpha_i^\vee})$.)

For any $0 \leq i \leq n$ and any $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$, each factor in $D_i(Y^{\alpha_i^\vee})D_i(Y^{-\alpha_i^\vee})|_\lambda$ is either (13) or (14) up to a monic Laurent monomial. We see that (13) or (14) is factorized in \mathcal{A} into (8), (9), respectively.

For any $0 \leq i \leq n$ and any $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$, each factor in $N_i(Y^{\alpha_i^\vee})N_i(Y^{-\alpha_i^\vee})|_\lambda$ is either of the following Laurent polynomials up to a monic Laurent monomial:

- (16) $t^{k+1}q^{r-1} - 1$ $(n \geq k+1 \geq 0, r-1 \geq 1), \text{ or}$
- (17) $t^{k+1}q^{r-1}a^{*2} - 1$ $(2n-2 \geq k+1 \geq 0, r-1 \geq 1), \text{ or}$
- (18) $t^{n-i}q^{r-1}a^*b^{*\pm 1} - 1$ $(n \geq i \geq 1, r-1 \geq 1), \text{ or}$
- (19) $t^{n-i}q^{r-1-\theta(\pm 1)}a^*c^{*\pm 1} - 1$ $(n \geq i \geq 1, r-1 \geq 1), \text{ or}$
- (20) $t^{n-i}q^{r-1-\theta(\pm 1)}a^*d^{*\pm 1} - 1$ $(n \geq i \geq 1, r-1 \geq 1),$

where $\theta(+1) = 1$ and $\theta(-1) = 0$. We see that (16), (17), (18), (19), or (20) is factorized in \mathcal{A} into (8), (9), (10), (11), or (12), respectively.

Suppose that specialized parameters are not roots of either of (8), ..., (12). Then we have

- (1) $N_i(Y^{\alpha_i^\vee})N_i(Y^{-\alpha_i^\vee})|_\lambda \neq 0$ for any $0 \leq i \leq n$ and $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$,
 - (2) $D_i(Y^{\alpha_i^\vee})D_i(Y^{-\alpha_i^\vee})|_\lambda \neq 0$ for any $0 \leq i \leq n$ and $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$,
 - (3) the specialized parameters are not any roots of either (13) or (14).
- (Hence from Lemma 3.2, the polynomial representation is Y -semisimple.)

Recall the definition of $c_{i,\lambda}$ in Proposition 2.5. We see that under the specialization of the parameters, $c_{i,\lambda}^{\pm 1}$ has no pole or zero for any $0 \leq i \leq n$ and $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$. Therefore any non-symmetric Koornwinder polynomial E_λ is cyclic and the polynomial representation is irreducible for the specialized parameter. \square

Hereafter, we will treat a specialization of parameters where the parameters are roots of one of the Laurent polynomials (8), (9), (10), (11), (12), but the parameters are not roots of the other Laurent polynomials. We call such a specialization *exclusive*.

Definition 3.3 (exclusive specialization of parameters). We realize an exclusive specialization of parameters as follows. Let $s \in \mathcal{A}$ be one of the irreducible factors of (8), (9), (10), (11), (12). Then $\mathcal{A}/s\mathcal{A}$ is an integral domain and we denote the fractional field of $\mathcal{A}/s\mathcal{A}$ by \mathbb{K}_s . We call s a *specialization polynomial*.

Definition 3.4 (order of zeros or poles). Let s be a specialization polynomial. For any $a \in \mathcal{A}$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $s^{-m}a \in \mathcal{A} \setminus s\mathcal{A}$. We denote this integer m by $\zeta_{s=0}(a)$, or simply by $\zeta(a)$. For any $c = a/b \in \mathbb{K}$ ($a, b \in \mathcal{A}$), define $\zeta(c) = \zeta(a) - \zeta(b)$. For any $f = \sum_\lambda c_\lambda x^\lambda \in P_n$, define $\zeta(f) = \min_\lambda \{\zeta(c_\lambda)\}$.

For any $a \in \mathcal{A}$, let \bar{a} be the quotient image in $\mathcal{A}/s\mathcal{A}$. For any $c = a/b$ such that $\zeta(c) \geq 0$, define $c|_{s=0} = \overline{s^{-\zeta(b)}a / s^{-\zeta(b)}b}$. For any $f = \sum_\lambda c_\lambda x^\lambda \in P_n$ such that $\zeta(f) \geq 0$, define $f|_{s=0} = \sum_\lambda c_\lambda|_{s=0} x^\lambda$.

Let \mathcal{H}_n^s and P_n^s be the (C_n^\vee, C_n) -DAHA and its polynomial representation over the field \mathbb{K}_s . Hereafter, unless stated otherwise, we express the elements in \mathbb{K}_s , \mathcal{H}_n^s , or P_n^s by the specialization map $|_{s=0}$ (or mentioning “at $s = 0$ ”). Without the specialization map $|_{s=0}$ (or without mentioning “at $s = 0$ ”), any scalars, operators, or Laurent polynomials are elements in \mathbb{K} , \mathcal{H}_n , or P_n .

3.2. Modified intertwiners and modified polynomials. For a specialization polynomial s and some $\lambda \in \mathbb{Z}^n$, the multiplicity of Y -eigenvalue $y(\lambda)|_{s=0}$ in P_n^s is possibly greater than 1, or E_λ possibly has poles at $s = 0$. We can cancel the poles by taking linear combinations of non-symmetric Koornwinder polynomials whose Y -eigenvalues are equal, but there are many choices for such combinations.

In this subsection, we introduce a linear combination \bar{E}_λ of the form

$$(21) \quad \bar{E}_\lambda = E_\lambda + \sum_{\mu} m_{\lambda\mu} \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu \quad (m_{\lambda\mu} \in \mathbb{Q}).$$

The sum runs over μ such that $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$. By taking suitable $m_{\lambda\mu}$, $(\bar{E}_\lambda)|_{s=0}$ has no pole at $s = 0$ (it may be a generalized Y -eigenvector). We call such a combination (21) a *modified polynomial*. We will give a basis of the polynomial representation P_n^s in terms of $(\bar{E}_\lambda)|_{s=0}$.

A sketch of construction of the basis is as follows.

For a given \bar{E}_λ , we define a modification $\bar{\phi}_i$ of the intertwiner ϕ_i . This $\bar{\phi}_i$ sends \bar{E}_λ to $c_{i,\lambda} \bar{E}_{s_i \cdot \lambda}$. We call it a *modified intertwiner*. The modified intertwiner gives an inductive and combinatorial way to compute the coefficients $m_{\lambda\mu}$ in the linear combination (21).

For any $\lambda \in \mathbb{Z}^n$, we take $w = s_{i_\ell} \cdots s_{i_1} \in W$ such that $\lambda = w \cdot (0, \dots, 0)$ and we obtain \bar{E}_λ from $E_{(0, \dots, 0)} = 1$ by acting modified intertwiners $\bar{\phi}_{i_1}, \dots, \bar{\phi}_{i_\ell}$. We will see that \bar{E}_λ is monic, and the terms in \bar{E}_λ are lower than x^λ with respect to the ordering \succ .

We note that the modified intertwiners $\bar{\phi}_i$ do not satisfy the braid relations though the original intertwiners ϕ_i satisfy them. Therefore, the coefficients $m_{\lambda\mu}$ in \bar{E}_λ (21) depend on reduced expressions of w above.

We state two lemmas. (They are corollaries of Section 2.)

Lemma 3.5. *For any $0 \leq i \leq n$ and any $\lambda \in \mathbb{Z}^n$, we have*

$$\begin{aligned} \frac{\chi_0^*(E_{s_i \cdot \lambda})}{\chi_0^*(E_\lambda)} c_{i,\lambda} &= \left. \frac{N_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})} \right|_\lambda \quad (1 \leq i \leq n-1) \\ \frac{\chi_0^*(E_{s_n \cdot \lambda})}{\chi_0^*(E_\lambda)} c_{n,\lambda} &= \left. \frac{N_n(Y^{\alpha_n^\vee})}{D_n(Y^{\alpha_n^\vee})} \right|_\lambda \\ \frac{\chi_0^*(E_{s_0 \cdot \lambda})}{\chi_0^*(E_\lambda)} c_{0,\lambda} &= \left. \frac{N_0(Y^{\alpha_0^\vee})}{D_0(Y^{\alpha_0^\vee})} \right|_\lambda. \end{aligned}$$

Lemma 3.6. *For any $0 \leq i \leq n$ and any $\lambda \in \mathbb{Z}^n$, we have*

$$\begin{aligned} T'_i &\stackrel{\text{def}}{=} T_i - t^{1/2} &= \phi_i - \frac{N_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})} \quad (1 \leq i \leq n-1) \\ T'_n &\stackrel{\text{def}}{=} T_n - t_n^{1/2} &= \phi_n - \frac{N_n(Y^{\alpha_n^\vee})}{D_n(Y^{\alpha_n^\vee})} \\ T'_0 &\stackrel{\text{def}}{=} U_n - u_n^{1/2} &= \phi_0 - \frac{N_0(Y^{\alpha_0^\vee})}{D_0(Y^{\alpha_0^\vee})}. \end{aligned}$$

Before giving an explicit definition of the modified intertwiners, we will show an example.

Example 3.7. Suppose $n = 4$. Let s be an irreducible factor in $t^2q - 1$, and $\lambda := (0, 1, 0, 1)$. We can easily check that $E_\lambda|_{s=0}$ is well-defined. There are many paths to generate another $\mu \in \mathbb{Z}^n$ from $\lambda = (0, 1, 0, 1)$ by applying $s_i \in W$. For instance,

$$\lambda = (0, 1, 0, 1) \xrightarrow{s_3} (0, 1, 1, 0) \xrightarrow{s_1} (1, 0, 1, 0) \xrightarrow{s_2} (1, 1, 0, 0) = s_2 s_1 s_3 \cdot \lambda.$$

In this example, we will construct polynomials $\bar{E}_{s_3 \cdot \lambda}$, $\bar{E}_{s_1 s_3 \cdot \lambda}$, $\bar{E}_{s_2 s_1 s_3 \cdot \lambda}$ from the given polynomial E_λ . We will see that they are of the form (21), well-defined at $s = 0$, and generalized Y -eigenfunctions at $s = 0$.

Since $(D_3(Y^{\alpha_3^\vee})|_\lambda)|_{s=0} = (t^2q - 1)|_{s=0} = 0$, the intertwiner ϕ_3 is not well-defined at $s = 0$. Put $\bar{\phi}_3 := T'_3 = T_3 - t^{1/2}$. Then we see that

$$\begin{aligned} \bar{\phi}_3 E_\lambda &= t^{1/2} E_{s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} t^{1/2} E_\lambda \\ &= t^{1/2} \bar{E}_{s_3 \cdot \lambda} \\ \text{where } \bar{E}_{s_3 \cdot \lambda} &:= E_{s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_\lambda. \end{aligned}$$

Since $\bar{\phi}_3|_{s=0}$ and $E_\lambda|_{s=0}$ are well-defined, $(\bar{E}_{s_3 \cdot \lambda})|_{s=0}$ is also well-defined. Moreover, from the commuting relations between $\bar{\phi}_3$ and Y_i , we see that $(\bar{E}_{s_3 \cdot \lambda})|_{s=0}$ is a generalized Y -eigenvector.

Since $(D_1(Y^{\alpha_1^\vee})|_{s_3 \cdot \lambda})|_{s=0} = (t^2q - 1)|_{s=0} = 0$, the intertwiner ϕ_1 is not well-defined at $s = 0$. Put $\bar{\phi}_1 := T'_1 = T_1 - t^{1/2}$. Then we see that

$$\begin{aligned} \bar{\phi}_1 \bar{E}_{s_3 \cdot \lambda} &= T'_1 \left(E_{s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_\lambda \right) \\ &= t^{1/2} \left(E_{s_1 s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_3 \cdot \lambda})} E_{s_3 \cdot \lambda} \right) \\ &\quad - \frac{\chi_0^*(E_{s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} t^{1/2} \left(E_{s_1 \cdot \lambda} - \frac{\chi_0^*(E_{s_1 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_\lambda \right). \end{aligned} \tag{22}$$

From the previous lemma, we have

$$(23) \quad \frac{\chi_0^*(E_{s_1 \cdot \lambda})}{\chi_0^*(E_\lambda)} t^{1/2} = \frac{N_1(Y^{\alpha_1^\vee})}{D_1(Y^{\alpha_1^\vee})}|_\lambda,$$

$$(24) \quad \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_3 \cdot \lambda})} t^{1/2} = \frac{N_1(Y^{\alpha_1^\vee})}{D_1(Y^{\alpha_1^\vee})}|_{s_3 \cdot \lambda}.$$

Since $Y^{\alpha_1^\vee}|_\lambda = Y^{\alpha_1^\vee}|_{s_3 \cdot \lambda}$, we see that (23)=(24). Hence

$$(22) = t^{1/2} \bar{E}_{s_1 s_3 \cdot \lambda}, \quad \text{where}$$

$$\bar{E}_{s_1 s_3 \cdot \lambda} := E_{s_1 s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_3 \cdot \lambda})} E_{s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_1 \cdot \lambda})} E_{s_1 \cdot \lambda} + \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_\lambda.$$

Since $\bar{\phi}_1|_{s=0}$ and $(\bar{E}_{s_3 \cdot \lambda})|_{s=0}$ are well-defined, $(\bar{E}_{s_1 s_3 \cdot \lambda})|_{s=0}$ is also well-defined. Moreover, from the commuting relations between $\bar{\phi}_1$ and Y_i , we see that $(\bar{E}_{s_1 s_3 \cdot \lambda})|_{s=0}$ is a generalized Y -eigenvector.

Let $\bar{\phi}_2 \in \mathcal{H}_n$ be as follows:

$$\bar{\phi}_2 = \phi_2 \frac{D_2(Y^{\alpha_2^\vee})}{D_2(Y^{\alpha_2^\vee})|_{s_1 s_3 \cdot \lambda}} \times \left(\frac{Y^{\alpha_2^\vee} - Y^{\alpha_2^\vee}|_\lambda}{Y^{\alpha_2^\vee}|_{s_1 s_3 \cdot \lambda} - Y^{\alpha_2^\vee}|_\lambda} - \frac{N_2(Y^{\alpha_2^\vee})|_{s_1 s_3 \cdot \lambda}}{N_2(Y^{\alpha_2^\vee})|_\lambda} \frac{Y^{\alpha_2^\vee} - Y^{\alpha_2^\vee}|_{s_1 s_3 \cdot \lambda}}{Y^{\alpha_2^\vee}|_\lambda - Y^{\alpha_2^\vee}|_{s_1 s_3 \cdot \lambda}} \right).$$

Note that $(D_2(Y^{\alpha_i^\vee})|_{s_1 s_3 \cdot \lambda})|_{s=0} = (tq - 1)|_{s=0} \neq 0$. Although $N_2(Y^{\alpha_2^\vee})|_\lambda = t^{-1/2}(t^{-2}q^{-1} - 1)$ and $Y^{\alpha_2^\vee}|_{s_1 s_3 \cdot \lambda} - Y^{\alpha_2^\vee}|_\lambda = tq - t^{-3}q^{-1}$ have poles at $s = 0$, we see that these poles are cancelled out in $\bar{\phi}_2$. Indeed,

$$\begin{aligned} & \boxed{\dots} \\ &= \frac{Y^{\alpha_2^\vee} - t^{-3}q^{-1}}{tq - t^{-3}q^{-1}} - \frac{t^2q - 1}{t^{-2}q^{-1} - 1} \frac{Y^{\alpha_2^\vee} - tq}{t^{-3}q^{-1} - tq} \\ & \text{(by putting } Z := t^2q), \\ &= \frac{Y^{\alpha_2^\vee} - t^{-1}Z^{-1}}{t^{-1}Z - t^{-1}Z^{-1}} - \frac{Z - 1}{Z^{-1} - 1} \frac{Y^{\alpha_2^\vee} - t^{-1}Z}{t^{-1}Z^{-1} - t^{-1}Z} \end{aligned}$$

and the factors $Z - 1$ in the denominators above are cancelled out. Thus we obtain that $\bar{\phi}_2|_{s=0} \in \mathcal{H}_n^s$ is well-defined. We have

$$\begin{aligned} & \bar{\phi}_2 \bar{E}_{s_1 s_3 \cdot \lambda} \\ &= t^{1/2} E_{s_2 s_1 s_3 \cdot \lambda} - c_{2, \lambda} \frac{D_2(Y^{\alpha_2^\vee})|_\lambda}{D_2(Y^{\alpha_2^\vee})|_{s_1 s_3 \cdot \lambda}} \frac{N_2(Y^{\alpha_2^\vee})|_{s_1 s_3 \cdot \lambda}}{N_2(Y^{\alpha_2^\vee})|_\lambda} \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_{s_2 \cdot \lambda} \\ &= t^{1/2} E_{s_2 s_1 s_3 \cdot \lambda} - c_{2, s_1 s_3 \cdot \lambda} \frac{\chi_0^*(E_{s_2 s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_1 s_3 \cdot \lambda})} \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_{s_2 \cdot \lambda})} \frac{\chi_0^*(E_{s_1 s_3 \cdot \lambda})}{\chi_0^*(E_\lambda)} E_{s_2 \cdot \lambda} \\ &= t^{1/2} \bar{E}_{s_2 s_1 s_3 \cdot \lambda} \\ & \text{where } \bar{E}_{s_2 s_1 s_3 \cdot \lambda} := E_{s_2 s_1 s_3 \cdot \lambda} - \frac{\chi_0^*(E_{s_2 s_1 s_3 \cdot \lambda})}{\chi_0^*(E_{s_2 \cdot \lambda})} E_{s_2 \cdot \lambda}. \end{aligned}$$

Since $\bar{\phi}_2|_{s=0}$ and $\bar{E}_{s_1 s_3 \cdot \lambda}|_{s=0}$ are well-defined, we see that $(\bar{E}_{s_2 s_1 s_3 \cdot \lambda})|_{s=0}$ is also well-defined. Moreover, from the commuting relations between $\bar{\phi}_2$ and Y_i , we see that $(\bar{E}_{s_2 s_1 s_3 \cdot \lambda})|_{s=0}$ is a generalized Y -eigenvector. \square

In this example, we have given the operators $\bar{\phi}_i \in \mathcal{H}_n$ and linear combinations of non-symmetric Koornwinder polynomials of the form (21). They are examples of *modified intertwiners* and *modified polynomials* which will be defined below. From now on, we introduce a general setting.

Recall the definition of $D_i(Y^{\alpha_i^\vee})$ and $N_i(Y^{\alpha_i^\vee})$ (see (6) and (7)). We easily see that $D_i(Y^{\alpha_i^\vee})|_\lambda \neq 0$ for any λ , and $\lambda = s_i \cdot \lambda \Leftrightarrow c_{i,\lambda} = 0 \Leftrightarrow N_i(Y^{\alpha_i^\vee})|_\lambda = 0$. Note that $D_i(Y^{\alpha_i^\vee})$ is the denominator of ϕ_i .

Now we define an element ϕ_i in \mathcal{H}_n .

Definition 3.8 (modified intertwiners). Let s be a specialization polynomial. Fix $0 \leq i \leq n$ and $\lambda \in \mathbb{Z}^n$ such that $\lambda \neq s_i \cdot \lambda$. Let $S_i = S_i(\lambda)$ be a finite set in \mathbb{Z}^n such that (i) $\lambda \in S_i$, (ii) $\mu \neq s_i \cdot \mu$ for any $\mu \in S_i$, (iii) $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ for any $\mu \in S_i$. For $\mu, \nu \in S_i$, we write $\mu \sim \nu$ if $Y^{\alpha_i^\vee}|_\mu = Y^{\alpha_i^\vee}|_\nu$. Let $\tilde{S}_i = S_i / \sim$, and $\tilde{\mu}$ be the equivalence class of $\mu \in S_i$. For $\mu \in S_i$, put

$$\begin{aligned} n_{\mu\lambda} &= \frac{N_i(Y^{\alpha_i^\vee})|_\mu}{N_i(Y^{\alpha_i^\vee})|_\lambda} \Big|_{s=0} \\ d_{\lambda\mu} &= \frac{D_i(Y^{\alpha_i^\vee})|_\lambda}{D_i(Y^{\alpha_i^\vee})|_\mu} \Big|_{s=0}. \end{aligned}$$

(Well-definedness of $n_{\mu\lambda}$ and $d_{\lambda\mu}$ is stated in Proposition 3.9 below.) Then, we denote the following element in \mathcal{H}_n by $\bar{\phi}_i(\lambda, \tilde{S}_i)$:

If $(D_i(Y^{\alpha_i^\vee})|_\lambda)|_{s=0} \neq 0$, then put

$$\bar{\phi}_i(\lambda, \tilde{S}_i) := \phi_i \frac{D_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})|_\lambda} \sum_{\tilde{\mu} \in \tilde{S}_i} n_{\mu\lambda} \frac{N_i(Y^{\alpha_i^\vee})|_\lambda}{N_i(Y^{\alpha_i^\vee})|_\mu} \prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee} - Y^{\alpha_i^\vee}|_\nu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu}$$

and if $(D_i(Y^{\alpha_i^\vee})|_\lambda)|_{s=0} = 0$, then put

$$\bar{\phi}_i(\lambda, \tilde{S}_i) := T'_i \sum_{\tilde{\mu} \in \tilde{S}_i} d_{\lambda\mu} \frac{D_i(Y^{\alpha_i^\vee})|_\mu}{D_i(Y^{\alpha_i^\vee})|_\lambda} \frac{N_i(Y^{\alpha_i^\vee})|_\lambda}{N_i(Y^{\alpha_i^\vee})|_\mu} \prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee} - Y^{\alpha_i^\vee}|_\nu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu}.$$

We call $\bar{\phi}_i(\lambda, \tilde{S}_i)$ a *modified intertwiner*.

We sometimes write them as $\bar{\phi}_i$ for simplicity. We will show well-definedness of $\bar{\phi}_i|_{s=0}$.

Proposition 3.9 (well-definedness). *Let s be a specialization polynomial.*

Then $n_{\mu\lambda} = \frac{N_i(Y^{\alpha_i^\vee})|_\mu}{N_i(Y^{\alpha_i^\vee})|_\lambda}|_{s=0}$ and $d_{\lambda\mu} = \frac{D_i(Y^{\alpha_i^\vee})|_\lambda}{D_i(Y^{\alpha_i^\vee})|_\mu}|_{s=0}$ are well-defined and belong to \mathbb{Q} . (Hence $\bar{\phi}_i \in \mathcal{H}_n$ is well-defined.) The specialized modified intertwiner $(\bar{\phi}_i)|_{s=0}$ is a well-defined element in \mathcal{H}_n^s .

In order to prove it, we use the following lemma.

Lemma 3.10. *Let s be a specialization polynomial. Then s is an irreducible Laurent polynomial in \mathcal{A} of the form $s = z - \omega$ where $z \in \mathcal{A}$ is a monic Laurent monomial and ω is a primitive ℓ -th root of unity for some ℓ . If $(z' - 1) \in s\mathcal{A}$ for a monic Laurent monomial $z' \in \mathcal{A}$, then $z' = z^{\ell m}$ for some $m \in \mathbb{Z}$.*

Proof of Proposition 3.9. In this proof, for simplicity, we write $N_i(Y^{\alpha_i^\vee})|_\lambda$ or $D_i(Y^{\alpha_i^\vee})|_\lambda$ as $N_i(\lambda)$ or $D_i(\lambda)$ for any $\lambda \in \mathbb{Z}^n$.

From Lemma 3.10 above, $s = z - \omega$ is an irreducible Laurent polynomial for a monic Laurent monomial z and a primitive ℓ -th root of unity for some ℓ .

First we show the well-definedness of $n_{\mu\lambda}$ for $i = 0$. (The proof for the other i and $d_{\lambda\mu}$ are similar.)

Since $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ for any $\mu \in S_0$, it holds that $N_0(\lambda)|_{s=0} = N_0(\mu)|_{s=0}$. Hence if $N_0(\lambda)|_{s=0} \neq 0$, then $n_{\mu\lambda} = 1$. Suppose that $N_0(\lambda)|_{s=0} = 0$. We have

$$\begin{aligned} N_0(\lambda) &= u_n^{1/2}(q^{1/2}y(\lambda)_1 - q^{1/2}c')(q^{1/2}y(\lambda)_1 - q^{1/2}d') \\ &= qc'd'u_n^{1/2}(c^*y(\lambda)_1 - 1)(d^*y(\lambda)_1 - 1). \end{aligned}$$

Hence a factor either $(c^*y(\lambda)_1 - 1)$ or $(d^*y(\lambda)_1 - 1)$ vanishes and the other factor does not vanish at $s = 0$. Assume that $(c^*y(\lambda)_1 - 1)|_{s=0} = 0$. Since $c^*y(\lambda)_1$ is a monic Laurent monomial, from Lemma 3.10, we see that $c^*y(\lambda) = z^{\ell m_1}$ for some $m_1 \in \mathbb{Z}$. Similarly, $c^*y(\mu) = z^{\ell m_2}$ for some $m_2 \in \mathbb{Z}$. Since $\lambda \neq s_0 \cdot \lambda$, we have $N_0(\lambda) \neq 0$ and $m_1 \neq 0$. Thus

$$(25) \quad n_{\mu\lambda} \stackrel{\text{def}}{=} \frac{N_0(\mu)}{N_0(\lambda)} \Big|_{s=0} = \frac{z^{\ell m_2} - 1}{z^{\ell m_1} - 1} \Big|_{s=0} = \frac{m_2}{m_1} \in \mathbb{Q}.$$

Next, we show the well-definedness for $\bar{\phi}_0$ for the case $D_0(\lambda)|_{s=0} = 0$. Proofs for the other operators are similar.

Take the representatives of \tilde{S}_0 by $\{\mu^{(1)} = \lambda, \mu^{(2)}, \mu^{(3)}, \dots\}$. Suppose $D_0(\lambda)|_{s=0} = 0$. Then $D_0(\mu^{(k)})|_{s=0} = 0$ for any k . Note that $D_0(\mu^{(k)}) = qy(\mu^{(k)})_1^2 - 1$ and $qy(\mu^{(k)})_1^2$ is a monic Laurent monomial. Moreover, the power of $qy(\mu^{(k)})_1^2$ with respect to the variables of $q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}$ are even numbers. Since $s = z - \omega$ is irreducible, the powers of z with respect to the variables $q^{1/2}, t^{1/2}, t_n^{1/2}, t_0^{1/2}, u_n^{1/2}, u_0^{1/2}$ contain at least one odd number. Hence from Lemma 3.10, $D_0(\mu^{(k)}) = z^{-2\ell m_k} - 1$ ($m_k \in \mathbb{Z}_{\neq 0}$). It means that $Y^{\alpha_0^\vee}|_{\mu^{(k)}} = z^{\ell m_k}$. Thus $N_0(\mu^{(k)})|_{s=0} \neq 0$ and $m_i \neq m_j$ (for any $i \neq j$). Therefore

$$\bar{\phi}_0 = \sum_{1 \leq k \leq |\tilde{S}_0|} \frac{2m_1}{2m_k} \frac{z^{-2\ell m_k} - 1}{z^{-2\ell m_1} - 1} \frac{N_0(z^{\ell m_1})}{N_0(z^{\ell m_k})} \prod_{1 \leq j \leq |\tilde{S}_0|, j \neq k} \frac{Y^{\alpha_0^\vee} - z^{\ell m_j}}{z^{\ell m_k} - z^{\ell m_j}}.$$

Note that $\zeta_{s=0}(N_0(z^{\ell m_k})) = 0$. Since $\bar{\phi}_0$ is a rational function with respect to z^ℓ , put $z^\ell = Z$. We obtain well-definedness of $\bar{\phi}_0|_{s=0}$ if $\bar{\phi}_0$ does not have

any poles at $Z = 1$. Reduce $\bar{\phi}_0$ to a common denominator and take Taylor expansions at $Z = 1$ of the numerator and the common denominator. Then we see that all factors $Z - 1$ in the denominator are cancelled out. \square

From Lemma 3.5 and 3.6, we easily obtain other expressions of the modified intertwiners: If $(D_i(Y^{\alpha_i^\vee})|_\lambda)|_{s=0} \neq 0$ then

$$(26) \quad \bar{\phi}_i(\lambda, \tilde{S}_i) = \sum_{\tilde{\mu} \in \tilde{S}_i} \left(n_{\mu\lambda} \frac{\chi_0^*(E_{s_i \cdot \lambda})}{\chi_0^*(E_\lambda)} \frac{\chi_0^*(E_\mu)}{\chi_0^*(E_{s_i \cdot \mu})} \frac{c_{i,\lambda}}{c_{i,\mu}} \phi_i \frac{D_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})|_\mu} \prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee}|_\nu - Y^{\alpha_i^\vee}|_\mu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu} \right),$$

and if $(D_i(Y^{\alpha_i^\vee})|_\lambda)|_{s=0} = 0$ then

$$(27) \quad \bar{\phi}_i(\lambda, \tilde{S}_i) = \sum_{\tilde{\mu} \in \tilde{S}_i} \left(d_{\lambda\mu} \frac{\chi_0^*(E_{s_i \cdot \lambda})}{\chi_0^*(E_\lambda)} \frac{\chi_0^*(E_\mu)}{\chi_0^*(E_{s_i \cdot \mu})} \frac{c_{i,\lambda} T'_i}{c_{i,\mu}} \prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee}|_\nu - Y^{\alpha_i^\vee}|_\mu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu} \right).$$

Definition 3.11 (modified polynomials). For a finite set S' satisfying $\lambda \notin S'$ and $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ ($\forall \mu \in S'$), denote the following sum by $\bar{E}_{\lambda, \{(m_{\lambda\mu}, \mu); \mu \in S'\}}$:

$$\bar{E}_{\lambda, \{(m_{\lambda\mu}, \mu); \mu \in S'\}} := E_\lambda + \sum_{\mu \in S'} m_{\lambda\mu} \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu \quad (m_{\lambda\mu} \in \mathbb{Q}).$$

We call it a *modified polynomial*. We denote the data $\{(m_{\lambda\mu}, \mu); \mu \in S'\}$ of modification terms by $\text{mt}(\lambda)$. For simplicity, we sometimes write $\bar{E}_{\lambda, \text{mt}(\lambda)}$ as \bar{E}_λ .

Proposition 3.12 (recurrence formula for modified polynomials). *For a finite set S' satisfying $\lambda \notin S'$ and $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ ($\forall \mu \in S'$), suppose that $(\bar{E}_{\lambda, \{(m_{\lambda\mu}, \mu); \mu \in S'\}})|_{s=0}$ is well-defined. Fix $0 \leq i \leq n$ and put $S_i := S' \setminus \{\mu \in S'; \mu = s_i \cdot \mu\}$ and $\text{mt}(\lambda) := \{(m_{\lambda\mu}, \mu); \mu \in S'\}$.*

(i) *If $\lambda \neq s_i \cdot \lambda$ and $D_i(\lambda)|_{s=0} \neq 0$ then*

$$\begin{aligned} \bar{\phi}_i(\lambda, \widetilde{S_i \cup \{\lambda\}}) \bar{E}_{\lambda, \text{mt}(\lambda)} &= c_{i,\lambda} \bar{E}_{s_i \cdot \lambda, \text{mt}(s_i \cdot \lambda)} \\ \text{where } \text{mt}(s_i \cdot \lambda) &:= \{(n_{\mu\lambda} m_{\lambda\mu}, s_i \cdot \mu); \mu \in S_i\} \end{aligned}$$

and the both sides are well-defined at $s = 0$.

(ii) *If $\lambda \neq s_i \cdot \lambda$ and $D_i(\lambda)|_{s=0} = 0$ then*

$$\begin{aligned} \bar{\phi}_i(\lambda, \widetilde{S_i \cup \{\lambda\}}) \bar{E}_{\lambda, \text{mt}(\lambda)} &= c_{i,\lambda} \bar{E}_{s_i \cdot \lambda, \text{mt}(s_i \cdot \lambda)} \\ \text{where } \text{mt}(s_i \cdot \lambda) &:= \{(-1, \lambda)\} \cup \{(d_{\lambda\mu} m_{\lambda\mu}, s_i \cdot \mu), (-d_{\lambda\mu} m_{\lambda\mu}, \mu); \mu \in S_i\} \end{aligned}$$

and the both sides are well-defined at $s = 0$.

(iii) Suppose $\lambda = s_i \cdot \lambda$. Then we see that $D_i(\lambda)|_{s=0} \neq 0$. For $\nu \in S'$, we have

$$\bar{\phi}_i(\nu, \tilde{S}_i) \bar{E}_{\lambda, \text{mt}(\lambda)} = c_{i, \nu} m_{\lambda \nu} \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\nu)} \bar{E}_{s_i \cdot \nu, \text{mt}(s_i \cdot \nu)}$$

$$\text{where } \text{mt}(s_i \cdot \nu) := \{(n_{\mu \nu} m_{\lambda \nu}^{-1} m_{\lambda \mu}, s_i \cdot \mu); \mu \in S_i\}$$

and the both sides are well-defined at $s = 0$.

Proof. We show (i). The proofs for (ii) and (iii) are similar. We see that

$$\phi_i \frac{D_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})|_\mu} \left(\prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee} - Y^{\alpha_i^\vee}|_\nu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu} \right) E_\mu = c_{i, \mu} E_{s_i \mu}, \text{ and}$$

$$\phi_i \frac{D_i(Y^{\alpha_i^\vee})}{D_i(Y^{\alpha_i^\vee})|_\mu} \left(\prod_{\tilde{\nu} \in \tilde{S}_i \setminus \{\tilde{\mu}\}} \frac{Y^{\alpha_i^\vee} - Y^{\alpha_i^\vee}|_\nu}{Y^{\alpha_i^\vee}|_\mu - Y^{\alpha_i^\vee}|_\nu} \right) E_{\nu'} = 0$$

(if $\tilde{\nu}' \in \tilde{S}_i \setminus \{\tilde{\mu}\}$ or $\nu' = s_i \cdot \nu'$).

Hence from the expression (26),

$$\begin{aligned} & \bar{\phi}_i(\lambda, \tilde{S}_i) \left(E_\lambda + \sum_{\mu \in S'} m_{\lambda \mu} \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu \right) \\ &= c_{i, \lambda} E_{s_i \lambda} + \sum_{\mu \in S_i} c_{i, \lambda} n_{\mu \lambda} m_{\lambda \mu} \frac{\chi_0^*(E_{s_i \cdot \lambda})}{\chi_0^*(E_{s_i \cdot \mu})} E_{s_i \cdot \mu} \end{aligned}$$

□

Remark 3.13. (i) For given $\lambda \in \mathbb{Z}^n$, $\text{mt}(\lambda)$, and $w \in W$, from Proposition 3.12, we can compute the following data inductively: $\text{mt}(s_{i_1} \cdot \lambda)$, $\text{mt}(s_{i_2} s_{i_1} \cdot \lambda)$, \dots , $\text{mt}(s_{i_\ell} \cdots s_{i_1} \cdot \lambda) = \text{mt}(w \cdot \lambda)$ where $w = s_{i_\ell} \cdots s_{i_1}$ is a reduced expression. However, in general, the result $\text{mt}(w \cdot \lambda)$ depends on reduced expressions of w . This fact corresponds to the fact that there are many choices for taking a basis of a generalized Y -eigenspace whose dimension $d \geq 2$ even except for a scalar multiple. (Recall Proposition 2.5. the (non-modified) intertwiners ϕ_i satisfy the braid relations. Hence $\phi_w := \phi_{i_\ell} \cdots \phi_{i_1}$ is well-defined. This fact corresponds to the fact that if multiplicity of Y -eigenvalue is 1, then the choice of a basis of the Y -eigenspace is unique up to a scalar multiple.)

(ii) For the DAHA of reduced affine root systems, Cherednik introduced chains of intertwiners and *non-semisimple Macdonald polynomials* in [Ch]. For (C_n^\vee, C_n) -DAHA, by applying the intertwiners ϕ_i and the operators T_i , one can also obtain *non-semisimple non-symmetric Koornwinder polynomials* (which are generalized Y -eigenfunctions). However, coefficients of lower terms in the non-semisimple polynomials will become more complicated than the modified polynomials defined above.

On the contrary, in this paper, we have defined rather complicated modified intertwiners. However, from Proposition 3.12, we obtain the data

$\text{mt}(s_i \cdot \lambda)$ of lower terms from a given $\text{mt}(\lambda)$ by appending data $n_{\mu\lambda}$ or $d_{\lambda\mu} \in \mathbb{Q}$. These rational numbers are nothing but ratios of powers of Y -eigenvalues (recall (25)). Consequently, the coefficients of lower terms and its computation become simpler.

Theorem 3.14 (construction of a basis). *Let $\lambda \in \mathbb{Z}^n$. Take the shortest element $w \in W$ such that $\lambda = w \cdot (0, \dots, 0)$. There exists a reduced expression $w = s_{j_\ell} \cdots s_{j_1}$ such that $\langle \lambda^{(m)}, \alpha_{j_m} \rangle > 0$ ($1 \leq m \leq \ell$) where $\lambda^{(m)} := s_{j_m} \cdots s_{j_1} \cdot (0, \dots, 0)$. Determine $\bar{E}_{\lambda^{(m)}}$ recursively by*

$$\bar{E}_{\lambda^{(m)}} := c_{j_m, \lambda^{(m-1)}}^{-1} \bar{\phi}_{j_m} \bar{E}_{\lambda^{(m-1)}}.$$

Then we have $\bar{E}_\lambda|_{s=0}$ is well-defined. The set $\{\bar{E}_\lambda|_{s=0}; \lambda \in \mathbb{Z}^n\}$ constitutes a \mathbb{K}_s -basis of the polynomial representation P_n^s .

Proof. By the well-definedness of $\bar{\phi}_{j_m}|_{s=0}$ and $\bar{E}_{\lambda^{(m-1)}}|_{s=0}$, we see that $(c_{j_m, \lambda^{(m-1)}} \bar{E}_{\lambda^{(m)}})|_{s=0}$ is well-defined. Since $\langle \lambda^{(m)}, \alpha_{j_m} \rangle > 0$, from Lemma 3.15 (see below), $c_{j_m, \lambda^{(m-1)}}$ is a monic Laurent monomial. Therefore $c_{j_m, \lambda^{(m-1)}}|_{s=0} \neq 0$ and $\bar{E}_{\lambda^{(m)}}|_{s=0}$ is well-defined.

From Lemma 3.15, the highest term in \bar{E}_λ is x^λ . Therefore $\{\bar{E}_\lambda|_{s=0}; \lambda \in \mathbb{Z}^n\}$ are linearly independent. \square

Lemma 3.15 (triangularity). *Assume that $m_{\lambda\mu} \neq 0$ for any $(m_{\lambda\mu}, \mu) \in \text{mt}(\lambda)$. We call $\text{mt}(\lambda)$ is triangular if $\mu \prec \lambda$ for any $(m_{\lambda\mu}, \mu) \in \text{mt}(\lambda)$. If $\langle s_i \cdot \lambda, \alpha_i \rangle > 0$, then we have*

- (i) $s_i \cdot \lambda \succ \lambda$, and $c_{i, \lambda}$ is a monic Laurent monomial,
- (ii) if $\text{mt}(\lambda)$ is triangular, then $\text{mt}(s_i \cdot \lambda)$ given by Proposition 3.12 is also triangular.

Proof. (i) is clear.

We will show (ii). Let $\text{mt}(s_i \cdot \lambda)$ be the data given in Proposition 3.12. For any $(m_{s_i \cdot \lambda, \nu}, \nu) \in \text{mt}(s_i \cdot \lambda)$, we see $\nu = \lambda, \mu$, or $s_i \cdot \mu$ for some $(m_{\lambda\mu}, \mu) \in \text{mt}(\lambda)$ such that $s_i \mu \neq \mu$. If $s_i \cdot \mu \prec \mu$, then $s_i \cdot \lambda \succ \lambda \succ \mu \succ s_i \cdot \mu$. If $s_i \cdot \mu \succ \mu$, then $s_i \cdot \lambda \succ s_i \cdot \mu$ because $s_i \cdot \lambda \succ \lambda$ and $\lambda \succ \mu$. In both cases, we have $s_i \cdot \lambda \succ s_i \cdot \mu$ and $s_i \cdot \lambda \succ \mu$. \square

Remark 3.16 (well-definedness test). Proposition 3.12 and Theorem 3.14 gives a sufficient condition for well-definedness of $E_\lambda|_{s=0}$ as follows: We have constructed a \mathbb{K}_s -basis $\{\bar{E}_{\lambda, \text{mt}(\lambda)}|_{s=0}; \lambda \in \mathbb{Z}^n\}$ of P_n^s . The construction is given by inductive (and combinatorial) computation of data $\text{mt}(\lambda)$ from $\text{mt}((0, \dots, 0)) = \emptyset$. If $\text{mt}(\lambda) = \emptyset$ for some $\lambda \in \mathbb{Z}^n$, then $\bar{E}_{\lambda, \text{mt}(\lambda)} = E_\lambda$ and we obtain the well-definedness of $E_\lambda|_{s=0}$.

4. POLYNOMIAL REPRESENTATIONS FOR SPECIALIZED PARAMETERS

In this section, we examine irreducibility and Y -semisimplicity of the polynomial representation under the exclusive specialization of parameters.

4.1. Arrow relation. We will investigate whether certain non-symmetric Koornwinder (or modified) polynomials are generated from other polynomials by actions of the (modified) intertwiners, or not. In this subsection, we introduce a notation for this purpose.

Definition 4.1 (arrow relation). *For given elements $\lambda, \lambda' \in \mathbb{Z}^n$ and data $\text{mt}(\lambda), \text{mt}(\lambda')$, suppose that two modified polynomials $\bar{E}_\lambda := E_{\lambda, \text{mt}(\lambda)}$ and $\bar{E}_{\lambda'} := E_{\lambda', \text{mt}(\lambda')}$ are well-defined at $s = 0$. Then we write*

$$(\lambda, \text{mt}(\lambda)) \rightarrow (\lambda', \text{mt}(\lambda'))$$

if there exist sequences $\lambda^{(0)} = \lambda, \lambda^{(1)}, \dots, \lambda^{(\ell)} = \lambda'$, and i_1, \dots, i_ℓ satisfying the following properties:

$$(\bar{\phi}_{i_m} \bar{E}_{\lambda^{(m-1)}})|_{s=0} = (c_m \bar{E}_{\lambda^{(m)}})|_{s=0} \neq 0$$

where $c_m \in \mathbb{K}$ and $\zeta(c_m) = 0$ for any $1 \leq m \leq \ell$.

We write

$$(\lambda, \text{mt}(\lambda)) \leftrightarrow (\lambda', \text{mt}(\lambda'))$$

if two arrow relations hold:

$$\begin{aligned} (\lambda, \text{mt}(\lambda)) &\rightarrow (\lambda', \text{mt}(\lambda')), \\ (\lambda', \text{mt}(\lambda')) &\rightarrow (\lambda, \text{mt}(\lambda)). \end{aligned}$$

For example, from Lemma 3.15, if $\langle s_i \cdot \lambda, \alpha_i \rangle > 0$ and $\text{mt}(\lambda)$ is triangular, then $(\lambda, \text{mt}(\lambda)) \rightarrow (s_i \cdot \lambda, \text{mt}(s_i \cdot \lambda))$ where $\text{mt}(s_i \cdot \lambda)$ is given in Proposition 3.12.

As a corollary of the previous section, we easily obtain the following arrow relation.

Proposition 4.2. *Suppose that $\zeta(N_i(\lambda)) = 0$, $\zeta(D_i(\lambda)) = 0$, and $\zeta(c_{i,\lambda}) = 0$. Take a modified polynomial $\bar{E}_\lambda := E_{\lambda, \{(m_{\lambda\mu}, \mu)\}_\mu}$ which is well-defined at $s = 0$. Then for the data $\{(m_{\lambda\mu}, s_i \cdot \mu)\}_\mu$ which is given in Proposition 3.12, the modified polynomial $\bar{E}_{s_i \cdot \lambda} := E_{s_i \cdot \lambda, \{(m_{\lambda\mu}, s_i \cdot \mu)\}_\mu}$ is well-defined at $s = 0$, and*

$$(\lambda, \{(m_{\lambda\mu}, \mu)\}_\mu) \rightarrow (s_i \cdot \lambda, \{(m_{\lambda\mu}, s_i \cdot \mu)\}_\mu).$$

4.2. The case $t^{k+1}q^{r-1} = 1$ ($k+1 \geq 2$). In this subsection, we treat the exclusive specialization of the case (8). Fix $n \geq k+1 \geq 2$ and $r-1 \geq 1$. Let $s \in \mathcal{A}$ be an irreducible factor of

$$t^{(k+1)/m} q^{(r-1)/m} - \omega_m$$

where $m = \text{GCD}(k+1, r-1)$ and ω_m is a primitive root of unity.

The purpose of the subsection is to realize a series of subrepresentations $I_1^{(k,r)} \subset I_2^{(k,r)} \subset \dots \subset I_{\lfloor \frac{n}{k+1} \rfloor}^{(k,r)}$ in the polynomial representation P_n^s as ideals defined by vanishing conditions. For $I_1^{(k,r)}$, we also show irreducibility and give a linear basis.

First we focus on $I_1^{(k,r)}$ and give a labeling set of its basis.

Definition 4.3 (neighborhood, admissibility). Fix integers (a, b) ($n \geq a \geq 2$, $b \geq 1$) and $\lambda \in \mathbb{Z}^n$. Put $w := w_\lambda^+$. The pair (i, j) is called a (a, b) -neighborhood if (i, j) satisfies three conditions as follows:

- (i) $|\rho(\lambda)_i| - |\rho(\lambda)_j| = a - 1$,
- (ii) $|\lambda_i| - |\lambda_j| \leq b$,
- (iii) if $|\lambda_i| - |\lambda_j| = b$, then,
 - (iii-1) $(\sigma(\lambda)_i, \sigma(\lambda)_j) = (+, +)$ and $i > j$, or
 - (iii-2) $(\sigma(\lambda)_i, \sigma(\lambda)_j) = (-, -)$ and $i < j$, or
 - (iii-3) $(\sigma(\lambda)_i, \sigma(\lambda)_j) = (-, +)$.

We denote the number of (a, b) -neighborhoods in λ by $\sharp^{(a,b)}(\lambda)$. If $\sharp^{(a,b)}(\lambda) = 0$, then λ is called (a, b) -admissible. Mostly we consider the case $(a, b) = (k+1, r-1)$. We sometimes omit (a, b) (we call it *type*) if it is clear.

Remark 4.4 (another definition of neighborhood, admissibility). For $\lambda \in \mathbb{Z}^n$, take the shortest element $w_\lambda^0 \in W_0$ such that $w_\lambda^0 \lambda \in \mathbb{Z}_{\geq 0}^n$. Define the map $\mathbb{Z}^n \rightarrow \mathbb{Z}_{\geq 0}^n$: $\lambda \mapsto \lambda^0 = w_\lambda^0 \lambda$. Then $\sharp^{(a,b)}(\lambda)$ is equal to the *number of neighborhoods of type (a, b) in λ^0* in the sense of [Ka2] (Definition 3.6). Especially, λ is admissible if and only if λ^0 has no neighborhoods of type (a, b) in the sense of [Ka2]. We will reduce some combinatorial arguments to those in [Ka2].

We will describe a basis of the irreducible subrepresentation in terms of the non-symmetric Koornwinder polynomials E_λ for some λ . The following proposition gives well-definedness of E_λ at $s = 0$.

Proposition 4.5. *For any λ such that $\sharp(\lambda) \leq 1$, there is no $\mu \in \mathbb{Z}^n$ such that $\mu \neq \lambda$ and $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ (resp. $y^*(\lambda)|_{s=0} = y^*(\mu)|_{s=0}$). As a corollary, E_λ (resp. E_λ^*) is well-defined at $s = 0$.*

Proof. If $y(\lambda) = y(\mu)$ at $s = 0$, then there exist integers $m_1, \dots, m_n \in \mathbb{Z}$ satisfying (i) $\lambda_i = \mu_i + (r-1)m_i$, (ii) $\rho(\lambda)_i = \rho(\mu)_i + (k+1)m_i$, and (iii) $\sigma(\lambda)_i = \sigma(\mu)_i$.

From (iii), we see that $w_\lambda^0 = w_\mu^0$. Hence there exist $m'_1, \dots, m'_n \in \mathbb{Z}$ (i') $\lambda_i^0 = \mu_i^0 + (r-1)m'_i$, (ii') $\rho(\lambda^0)_i = \rho(\mu^0)_i + (k+1)m'_i$, and (iii') $\sigma(\lambda^0)_i = \sigma(\mu^0)_i$. We also see that $\sharp(\lambda^0) = \sharp(\lambda)$.

For any element $\nu \in \mathbb{Z}_{\geq 0}^n$, the pair of integers $\rho(\nu)$ given in this paper and the pair of half-integers $\rho(\nu)$ given in [Ka2] (§2.3) differ only by the total shift $\frac{n-1}{2}$. In [Ka2], Lemma 4.13, by a combinatorial argument, it was shown that there is no $\mu^0 \neq \lambda^0$ satisfying (i'), (ii'), and $\sharp(\lambda^0) \leq 1$.

Therefore there is no $\mu \neq \lambda$ such that $y(\lambda) = y(\mu)$ at $s = 0$. The proof for the dual polynomial is similar. \square

Now we give one of the main theorems in this subsection.

Theorem 4.6. *The space $I_1^{(k,r)'} = \text{span}_{\mathbb{K}_s} \{E_\lambda|_{s=0}; \lambda \text{ is admissible}\}$ is an irreducible representation.*

Proof. By the definition, $I_1^{(k,r)'} is closed under the action of Y_1, \dots, Y_n . From Lemma 4.7 and Lemma 4.8 below, $I_1^{(k,r)'}$ is closed under the action of ϕ_0, \dots, ϕ_n . Hence $I_1^{(k,r)'}$ is closed under the action of \mathcal{H}_n^s .$

Take any non-zero $v \in I_1^{(k,r)'}$. Then there exists an admissible λ and $h \in \mathcal{H}_n^s$ such that $E_\lambda = hv$. Take another admissible μ . From Lemma 4.9 and Lemma 4.10 below, we have $\lambda \leftrightarrow \lambda^0$, $\mu \leftrightarrow \mu^0$, and $\lambda^0 \leftrightarrow (nM, \dots, 2M, M) \leftrightarrow \mu^0$ for a large enough M . Therefore any non-zero $v \in I_1^{(k,r)'}$ is cyclic. \square

Four steps of the proof of Theorem 4.6 are given as follows.

Lemma 4.7. *If λ is admissible and $s_i \cdot \lambda$ is not admissible for some $0 \leq i \leq n$, then $(\phi_i E_\lambda)|_{s=0} = 0$.*

Proof. From the assumption, $\sharp(\lambda) = 0$ and $\sharp(s_i \cdot \lambda) \geq 1$. Recall the defining condition (i), (ii), (iii-a), (iii-b), (iii-c) of neighborhood in Definition 4.3. If (j_1, j_2) is a neighborhood in $s_i \cdot \lambda$ for $i = 0$ or n , then (j_1, j_2) is also a neighborhood in λ . Therefore we have $i \neq 0$ and $i \neq n$.

For $1 \leq i \leq n-1$, if (j_1, j_2) is a neighborhood in $s_i \cdot \lambda$, then $(s_i(j_1), s_i(j_2))$ is also a neighborhood in λ except for the cases $(j_1, j_2) = (i, i+1)$ or $(i, i+1)$. Since there is no neighborhood in λ , the only neighborhood in $s_i \cdot \lambda$ is one of $(i, i+1)$ or $(i+1, i)$.

If $(i, i+1)$ is a neighborhood in $s_i \cdot \lambda$, then $(\sigma(s_i \cdot \lambda)_i, \sigma(s_i \cdot \lambda)_{i+1}) = (-, -)$. If $(i+1, i)$ is a neighborhood in $s_i \cdot \lambda$, then $(\sigma(s_i \cdot \lambda)_i, \sigma(s_i \cdot \lambda)_{i+1}) = (+, +)$. In both cases, $\langle \rho(s_i \cdot \lambda), \alpha_i \rangle = -k$ and $\langle s_i \cdot \lambda, \alpha_i \rangle = -(r-1)$. Hence $((t^{1/2} Y^{\alpha_i} - t^{-1/2})|_\lambda)|_{s=0} = 0$, and $c_{i,\lambda}|_{s=0} = 0$. Thus $(\phi_i E_\lambda)|_{s=0} = (c_{i,\lambda} E_{s_i \cdot \lambda})|_{s=0} = 0$. \square

Lemma 4.8. *If λ and $s_i \cdot \lambda$ are admissible for some $0 \leq i \leq n$, then $\lambda \leftrightarrow s_i \cdot \lambda$.*

Proof. It is easy to see that $\zeta(N_0(\lambda)) = \zeta(D_0(\lambda)) = \zeta(c_{0,\lambda}) = 0$ and $\zeta(D_n(\lambda)) = 0$. We see that $\zeta(c_{n,\lambda}) \neq 0$ or $\zeta(N_n(\lambda)) \neq 0$ only if

$$(28) \quad \lambda_n = (r-1)m \text{ and } \rho(\lambda)_n = (k+1)m \quad \text{for some } m \in \mathbb{Z}.$$

If (28) holds, then there exists a $((k+1)m+1, (r-1)m)$ -neighborhood in λ , and consequently there exists a $(k+1, r-1)$ -neighborhood in λ (see [Ka2], Lemma 4.7.) However it is inconsistent with the admissibility of λ . Therefore from Proposition 4.2, $\lambda \leftrightarrow s_i \cdot \lambda$ for $i = 0$ and n .

For $1 \leq i \leq n-1$, we see that $\zeta(N_i(\lambda)) \neq 0$ or $\zeta(D_i(\lambda)) \neq 0$ or $\zeta(c_{i,\lambda}) \neq 0$ only if

$$(29) \quad \begin{aligned} \langle \lambda, \alpha_i \rangle &= (r-1)m, \text{ and} \\ \langle \rho(\lambda), \alpha_i \rangle &= (k+1)m + d, \text{ and} \\ (\sigma(\lambda)_i, \sigma(\lambda)_{i+1}) &= (+, +) \text{ or } (-, -) \\ &\text{for some } m \in \mathbb{Z} \text{ and } d = -1, 0, 1. \end{aligned}$$

If (29) holds, then there exists a $((k+1)m+d+1, (r-1)m)$ -neighborhood in λ , and consequently there exists a $(k+1, r-1)$ -neighborhood in λ (see

[Ka2], Lemma 4.7.) However it is inconsistent with the admissibility of λ . Therefore from Proposition 4.2, $\lambda \leftrightarrow s_i \cdot \lambda$ for $1 \leq i \leq n-1$. \square

Lemma 4.9. *For any admissible λ , we have $\lambda \leftrightarrow \lambda^0$, where λ^0 is defined in Remark 4.4.*

Proof. For a reduced expression of $w_\lambda^0 = s_{i_l} \cdots s_{i_1}$, put $\lambda^{(j)} = s_{i_j} \cdots s_{i_1} \cdot \lambda$ ($1 \leq j \leq l$). We can easily check that $\lambda^{(j)}$ is admissible for any $1 \leq j \leq l$. Hence from Lemma 4.8, we have $\lambda^{(j-1)} \leftrightarrow \lambda^{(j)}$ for any $1 \leq j \leq l$. \square

Lemma 4.10. *For an admissible $\lambda \in \mathbb{Z}_{\geq 0}^n$ and a large enough M , define $\lambda^{j,M} \in \mathbb{Z}_{\geq 0}^n$ by $\lambda_i^{j,M} = \lambda_i$ if $\rho(\lambda)_i < n-j$ and $\lambda_i^{j,M} = (\rho(\lambda)_i + 1)M$ if $\rho(\lambda)_i \geq n-j$. Then we have $\lambda^{j,M} \leftrightarrow \lambda^{j+1,M}$.*

Proof. Take shortest element $w \in W$ such that $\lambda^{j+1,M} = w\lambda^{j,M}$ and take a reduced expression $w = s_{i_l} \cdots s_{i_1}$. Put $\lambda^{(m)} = s_{i_l} \cdots s_{i_1} \lambda^{j,M}$. Then we see that $\lambda^{(m)}$ is admissible for any $0 \leq m \leq l$. Hence from Lemma 4.8, $\lambda^{j,M} \leftrightarrow \lambda^{j+1,M}$. \square

We will give a series of subrepresentations $I_1^{(k,r)} \subset I_2^{(k,r)} \subset \cdots \subset I_{\lfloor \frac{n}{k+1} \rfloor}^{(k,r)}$ in terms of vanishing conditions for Laurent polynomials. We will show that the irreducible representation $I_1^{(k,r)'} \text{ coincides with } I_1^{(k,r)}$.

Definition 4.11 (m -wheel condition). Let $\{i_1, \dots, i_{k+1}\}$ be distinct indexes in $\{1, \dots, n\}$ and $\{\sigma_1, \dots, \sigma_{k+1}\}$ be a set of signs $+$ or $-$. For $m \in \mathbb{Z}/(k+1)\mathbb{Z}$, we denote by $\sigma_m i_m \mapsto \sigma_{m+1} i_{m+1}$ the constraint $z_{i_m}^{\sigma_m} t q^{p_m}|_{s=0} = z_{i_{m+1}}^{\sigma_{m+1}}$ for some $p_m \in \mathbb{Z}$ satisfying (i) and (ii):

- (i) $p_m \geq 0$
- (ii) $p_m = 0 \Rightarrow (\sigma_m, \sigma_{m+1}) = (+, +)$ and $i_m < i_{m+1}$, or
 $(\sigma_m, \sigma_{m+1}) = (+, -)$, or
 $(\sigma_m, \sigma_{m+1}) = (-, -)$ and $i_m > i_{m+1}$.

We call a closed cycle of the arrows “ \mapsto ” with length $k+1$ a *wheel*. (Since $(t^{k+1}q^{r-1} - 1)|_{s=0}$, we see that $p_1 + \cdots + p_{k+1} = r-1$.) The m -wheel condition for f is given as follows: $f(z) = 0$ if z_1, \dots, z_n form any disjoint m wheels.

Some examples for Laurent polynomials satisfying the 1-wheel condition is given in §4.3.

Lemma 4.12. *The 1-wheel condition for $f \in P_n^s$ is equivalent to the following vanishing condition: $\chi_\mu^*(f) = 0$ for any μ such that $\sharp(\mu) = 1$.*

Proof. Take a neighborhood (i, i') in μ . For $1 \leq \ell \leq k+1$, take i_ℓ and σ_ℓ satisfying $|\rho(\mu)_{i_\ell}| = |\rho(\mu)_i| - \ell + 1$ and $\sigma_\ell = \sigma(\mu)_{i_\ell}$. Then $\sigma_\ell i_\ell \mapsto \sigma_{\ell+1} i_{\ell+1}$ and the cycle of the arrows forms a wheel.

Conversely, since f is a Laurent polynomial, it is possible to replace the vanishing condition for the constraints $\sigma_m i_m \mapsto \sigma_{m+1} i_{m+1}$ by the vanishing condition for finitely many points in \mathbb{K}_s^n . We can realize such points as $(\chi_\mu^*(x_1)|_{s=0}, \dots, \chi_\mu^*(x_n)|_{s=0})$ for some μ satisfying $\sharp(\mu) = 1$. \square

Proposition 4.13. *The space $I_m^{(k,r)}$ of Laurent polynomials satisfying the m -wheel condition is a subrepresentation.*

Proof. Since the space $I_m^{(k,r)}$ is defined by the vanishing condition, invariance under the action of $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ is clear.

We show invariance under the action of T_0, \dots, T_n . Let A and B are coefficients of s_i and 1 in T_i : $T_i = As_i + B$. Take $f \in I_m^{(k,r)}$ and fix a wheel: a set of constraints $w = \{\sigma_m i_m \mapsto \sigma_{m+1} i_{m+1}\}_m$. Then Bf vanishes under w .

The constraint w for $s_i f$ is given by $s_i(z_{i_m})^{\sigma_m} t q^{p_m}|_{s=0} = s_i(z_{i_{m+1}})^{\sigma_{m+1}}$. Hence it is equivalent to $w' = \{s_i(\sigma_m i_m) \mapsto s_i(\sigma_{m+1} i_{m+1})\}_m$ provided that the powers of q in the new constraints w' satisfies the condition (i) and (ii) in Definition 4.11. If (i) or (ii) is invalid, then we see that A vanishes under w . Hence $As_i f$ vanishes under w . \square

Now we give the second main statement in this subsection.

Theorem 4.14. $I_1^{(k,r)} = I_1^{(k,r)'}.$

The relation $I_1^{(k,r)} \supset I_1^{(k,r)'}$ is shown in Proposition 4.15 below.

Proposition 4.15. *If λ is admissible, that is $\sharp(\lambda) = 0$, then $\zeta(\chi_0^*(E_\lambda)) = \lfloor \frac{n}{k+1} \rfloor$ and E_λ satisfies the wheel condition. If λ satisfies $\sharp(\lambda) = 1$, then $\zeta(\chi_0^*(E_\lambda^*)) = \lfloor \frac{n}{k+1} \rfloor - 1$.*

Proof. We see that

$$\begin{aligned} \zeta(\chi_0^*(E_\lambda)) &= \zeta(\chi_0(E_\lambda^*)) \\ &= \left\lfloor \frac{n}{k+1} \right\rfloor \\ &\quad - \sum_m (\text{the number of } ((k+1)m, (r-1)m)\text{-neighborhoods in } \lambda) \\ &\quad + \sum_m (\text{the number of } ((k+1)m+1, (r-1)m)\text{-neighborhoods in } \lambda). \end{aligned}$$

If λ is admissible, then the second and the third term is zero. If $\sharp(\lambda) = 1$, then the second term is 1 and the third term is zero.

Suppose that λ is admissible and $\sharp(\mu) = 1$. By the duality relation, we have

$$\chi_\mu^*(E_\lambda) = \frac{\chi_\lambda(E_\mu^*)}{\chi_0(E_\mu^*)} \chi_0^*(E_\lambda).$$

From Proposition 4.5 and the first half of this proof, we have $\zeta(\chi_\mu^*(E_\lambda)) \geq 1$. Hence from the argument in the proof of Lemma 4.12, we see that E_λ satisfy the wheel condition. \square

The rest of the proof of Theorem 4.14 is similar to Section 5 in [Ka2]. Consider a degree-restricted subspace of $I_1^{(k,r)}$, introduce a dual space of the subspace. By giving an ordering on monomials, obtain an upper bound of the dimension of the dual space. We will see that the upper bound coincides with the lower bound which is given in Proposition 4.15. \square

4.3. Examples of polynomials satisfying the wheel condition. In this subsection, we give two examples of *factorized* polynomials satisfying the 1-wheel condition (recall Definition 4.11) in the case $t^{k+1}q = 1$ and $t^{k+1}q^k = 1$.

Here we omit the specializing map $|_{s=0}$ for simplicity. (All parameters are regarded as elements in \mathbb{K}_s .)

Proposition 4.16. *Suppose that the parameters satisfy $t^{k+1}q = 1$ ($n \geq k+1 \geq 2$) and that $n = km$. Then*

$$E_{(m-1, \dots, 1, 0)^k} = \prod_{\ell=1}^k \prod_{m(\ell-1) < i < j \leq m\ell} (x_i - t^{-1}x_j) \left(1 - \frac{t^\ell q}{x_i x_j}\right).$$

Proof. Step 1 Let us show that the RHS satisfies the wheel condition.

Consider a cycle of $k+1$ arrows. Since $r-1=1$, the number of arrows from $-$ to $+$ is at most one. Hence the following three cases are all of possibilities: (i) all the signs are $+$, (ii) all the signs are $-$, (iii) the cycle is divided into one $+$ part and one $-$ part. In the case (i) and (ii), the factor $(x_i - t^{-1}x_j)$ in the RHS vanishes. Hereafter we assume (iii). In this case, note that the power of q in the arrow from $-$ to $+$ is 1 and the powers of q in the other arrows are 0.

The indexes $\{1, \dots, n\}$ are divided into k blocks: $\{1, \dots, m\}, \dots, \{n-m+1, \dots, n\}$. We enumerate these blocks from the left hand side. (e.g. $\{n-m+1, \dots, n\}$ is k -th block.) In this proof we mean by $+\rightarrow+$ the arrow $+i \mapsto +j$ with $i < j$, and by $-\leftarrow-$ the arrow $-i \mapsto -j$ with $i > j$. If three vertices of the wheel are in the same block, then there exists a pair of $+\rightarrow+$ or $-\leftarrow-$ and the factor $(x_i - t^{-1}x_j)$ vanishes. So we assume there are at most two vertices in each block.

Let us show the following claim by induction: “There exists ℓ -th block such that ℓ blocks from the first to ℓ -th blocks contain $\ell+1$ vertices of the wheel.”

Consider the first block. By assumption, it contains at most two vertices. If the number of vertices is two, then the claim holds. We assume that it contains at most one vertex.

Suppose that ℓ blocks from the first to ℓ -th blocks contain at most ℓ vertices. Then consider $\ell+1$ -th block. Since this block contains at most two vertices, $\ell+1$ blocks from the first to $\ell+1$ -th blocks contain: (a) at

most $\ell + 1$ vertices, or (b) $\ell + 2$ vertices. (b) implies the claim. If we are in the case (a), consider the next block.

Note that the number of blocks are finite, and that there are totally k blocks and $k + 1$ vertices. So the case (b) does occur at least once, and the claim is proved.

Consider the ℓ -th block in the claim. If ℓ -th block contains $(+, +)$ or $(-, -)$, then the factor $(x_i - t^{-1}x_j)$ vanishes. The remaining possibility is that ℓ -th block contains $+$ and $-$. Denote these two indexes by i_+ and i_- . From (iii), we see that the sequence of arrows from i_- to i_+ consists of one arrow from $-$ to $+$ and the other arrows are of the form $- \leftarrow -$ or $+ \rightarrow +$. Moreover, from the claim, the arrows from i_- to i_+ intertwines $\ell + 1$ indexes in ℓ blocks from the first to ℓ -th block. Hence the factor $(1 - \frac{t^l q}{x_{i_+} x_{i_-}})$ vanishes.

Step 2 Let $\lambda = (m - 1, \dots, 1, 0, \dots, m - 1, \dots, 1, 0, \dots, m - 1, \dots, 1, 0)$ (repeated by k times) and

$$S = \{ \text{admissible elements} \} \cap \{ \mu \in \mathbb{Z}^n; \mu \preceq \lambda \}.$$

Then we can check that $\#S = 1$.

Step 3 From Step 1, the RHS is written of the form $\sum_{\mu \in S} c_\mu E_\mu$. From Step 2, the RHS is equal to $c_\lambda E_\lambda$. Since the coefficient of x^λ in each side is 1, we have $c_\lambda = 1$. \square

Example 4.17. Suppose that the parameters satisfy $t^{k+1}q^k = 1$ and that $n = km$. Then, the product

$$\prod_{1 \leq i < j \leq n} (x_i x_j^{-1} - t^{-1})(x_j - t^{-1}x_i^{-1})$$

satisfies the wheel condition.

Proof. There exists at least one arrow whose power is 0. If the arrow is of the form $+ \rightarrow +$ or $- \leftarrow -$, then the factor $(x_i x_j^{-1} - t^{-1})$ vanishes. If the arrow is from $+$ to $-$, the factor $(x_j - t^{-1}x_i^{-1})$ vanishes. \square

4.4. Preliminaries for §4.5 and §4.6. In §4.5 and §4.6 below, we will show that the polynomial representation for specialized parameters P_n^s is irreducible. The proof is organized by the following three steps:

Step Irr-1 Define “large enough” elements in \mathbb{Z}^n . For any non-zero Laurent polynomial $f \in P_n^s$, show that $hf = E_\lambda|_{s=0}$ for an element $\exists h \in \mathcal{H}_n^s$ and a large enough element $\lambda \in \mathbb{Z}^n$.

Step Irr-2 Find a “specific element” $\lambda \in \mathbb{Z}^n$ such that $\mu \leftrightarrow \lambda$ for any large enough element $\mu \in \mathbb{Z}^n$.

Step Irr-3 For the specific element $\lambda \in \mathbb{Z}^n$ in Step 2, show that $\lambda \rightarrow (0, \dots, 0)$. Since $E_{(0, \dots, 0)} = 1$ is a cyclic vector in P_n^s , we obtain irreducibility of P_n^s .

In Step Irr-3, we will use the following arrow relations.

Proposition 4.18. *Suppose that $\langle s_i \cdot \lambda, \alpha_i \rangle > 0$, $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$, $E_{s_i \cdot \lambda}$ and $E_{\lambda, (-1, \mu)}$ and E_μ are well-defined at $s = 0$, $\zeta(D_i(s_i \cdot \lambda)) = 0$, $\zeta(N_i(s_i \cdot \lambda)) = 0$, $\zeta(\chi_0^*(E_\lambda)) - \zeta(\chi_0^*(E_\mu)) = -1$, and $\zeta(c_{i, \lambda}) = 1$. Then we have*

$$s_i \cdot \lambda \rightarrow \mu,$$

namely, $\bar{\phi}_i E_{s_i \cdot \lambda}|_{s=0} = \exists c E_\mu|_{s=0}$ where $\zeta(c) = 0$.

Proof. We have

$$\begin{aligned} \bar{\phi}_i E_{s_i \cdot \lambda} &= c_{i, \lambda} E_\lambda \\ &= c_{i, \lambda} \left(E_{\lambda, (-1, \mu)} + \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu \right). \end{aligned}$$

The first term vanishes at $s = 0$ and the second term is of the form $c E_\mu$ where $\zeta(c) = 0$. \square

Proposition 4.19. *Suppose that $\langle \lambda, \alpha_i \rangle = 0$, $\langle \mu, \alpha_i \rangle > 0$, $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$, $E_{\lambda, (-1, \mu)}$ and $E_{s_i \cdot \mu}$ are well-defined at $s = 0$, $\zeta(D_i(\lambda)) = 0$, $\zeta(\chi_0^*(E_\lambda)) - \zeta(\chi_0^*(E_\mu)) = -1$, and $\zeta(c_{i, \mu}) = 1$. Then we have*

$$(\lambda, (-1, \mu)) \rightarrow s_i \cdot \mu,$$

namely, $\phi_i E_{\lambda, (-1, \mu)}|_{s=0} = \exists c E_{s_i \cdot \mu}|_{s=0}$ where $\zeta(c) = 0$.

Proof. We have

$$\phi_i E_{\lambda, (-1, \mu)} = -\frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} c_{i, \mu} E_{s_i \cdot \mu}.$$

Denote the coefficient of $E_{s_i \cdot \mu}$ by c , then $\zeta(c) = 0$. \square

We will show well-definedness of certain non-symmetric Koornwinder polynomials and modified polynomials at $s = 0$ by the following argument.

Let $f \in P_n$ be a given Laurent polynomial. Then $f|_{s=0}$ is well-defined (namely, $\zeta_{s=0}(f) \geq 0$) if there exists a subset $N \subset \mathbb{Z}^n$ satisfying the following two conditions: for a large enough integer $M \gg \max_i(|\deg_{x_i} f|)$,

(Grid-i): $\{(\chi_\nu^*(x_1)|_{s=0}, \dots, \chi_\nu^*(x_n)|_{s=0}); \nu \in N\}$ contains distinct $M^n = M \times M \times \dots \times M$ points in \mathbb{K}_s^n ,

(Grid-ii): $\zeta_{s=0}(\chi_\nu^*(f)) \geq 0$ for any $\nu \in N$.

From this point of view, we will show the well-definedness of E_λ as follows:

Proposition 4.20. *For a given element $\lambda \in \mathbb{Z}^n$, suppose that there exists a subset $N \subset \mathbb{Z}^n$ satisfying (Grid-i) for a large enough $M \gg \max_i |\lambda_i|$ and*

(Grid-iii): E_ν^* has no pole at $s = 0$ for any $\nu \in N$,

(Grid-iv): $\zeta(\chi_0(E_\nu^*)) = \zeta(\chi_0^*(E_\lambda))$ for any $\nu \in N$.

Then E_λ has no pole at $s = 0$.

Proof. For any $\nu \in N$, from the duality relation,

$$\chi_\nu^*(E_\lambda) = \frac{\chi_0^*(E_\lambda)}{\chi_0(E_\nu^*)} \chi_\lambda(E_\nu^*).$$

Thus $\zeta(\chi_\nu^*(E_\lambda)) \geq 0$. It implies that E_λ has no pole. \square

Similarly, we will show the well-definedness of a modified polynomial as follows:

Proposition 4.21. *For given elements $\lambda, \mu \in \mathbb{Z}^n$ suppose that $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$, and that there exists a subset $N \subset \mathbb{Z}^n$ satisfying (Grid-i) for a large enough $M \gg \max_i \{|\lambda_i|, |\mu_i|\}$, (Grid-iii), and*

(Grid-v): $\zeta(\chi_0(E_\nu^*)) = \zeta(\chi_0^*(E_\lambda)) + 1$ for any $\nu \in N$.

Then $E_{\lambda,(-1,\mu)} = E_\lambda - \frac{\chi_0^(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu$ has no pole at $s = 0$.*

Proof. Take any $\nu \in N$. Then

$$\begin{aligned} & \chi_\nu^*(E_{\lambda,(-1,\mu)}) \\ &= \chi_\nu^*(E_\lambda) - \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} \chi_\nu^*(E_\mu) \\ &= \chi_0^*(E_\lambda) \left(\frac{\chi_\nu^*(E_\lambda)}{\chi_0^*(E_\lambda)} - \frac{\chi_\nu^*(E_\mu)}{\chi_0^*(E_\mu)} \right) \\ &= \frac{\chi_0^*(E_\lambda)}{\chi_0(E_\nu^*)} (\chi_\lambda(E_\nu^*) - \chi_\mu(E_\nu^*)) \quad (\text{duality}). \end{aligned}$$

Since $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$, we have $\zeta(\chi_\lambda(E_\nu^*) - \chi_\mu(E_\nu^*)) \geq 1$. Thus $\chi_\nu^*(E_{\lambda,(-1,\mu)})$ has no pole at $s = 0$. It implies that $E_{\lambda,(-1,\mu)}$ has no pole at $s = 0$. \square

4.5. The case $tq^{r-1} = 1$. Fix $r - 1 \geq 1$ and let $s \in \mathcal{A}$ be an irreducible factor of

$$tq^{r-1} - 1.$$

The main theorem of this subsection is as follows:

Theorem 4.22. *The polynomial representation P_n^s is irreducible.*

We follow the sketch of proof given in §4.4.

4.5.1. Step Irr-1.

Definition 4.23 (large enough elements). In this subsection, $\lambda \in \mathbb{Z}^n$ is called *large enough* if λ satisfies

$$\lambda = \lambda^+ \text{ and } \lambda_i - \lambda_{i+1} \geq 2(r-1).$$

Lemma 4.24. *Take any large enough λ . Then there are no $\mu \neq \lambda$ satisfying $y(\mu) = y(\lambda)$ or $y^*(\mu) = y^*(\lambda)$ at $s = 0$. As a corollary, E_λ and E_λ^* have no pole at $s = 0$.*

Proof. Take μ satisfying $y(\mu) = y(\lambda)$ or $y^*(\mu) = y^*(\lambda)$ at $s = 0$. Then there exist integers m_1, \dots, m_n such that $\mu_i = \lambda_i + (r-1)m_i$, $\rho(\mu)_i = \rho(\lambda)_i + m_i$, and $\sigma(\mu)_i = \sigma(\lambda)_i$. However there is no such μ except for $\mu = \lambda$. \square

Lemma 4.25. *Let f be any non-zero Laurent polynomial. Then there exists large enough λ such that $E_\lambda|_{s=0} \in \mathcal{H}_n^s f$.*

Proof. Let f be any Laurent polynomial. Take $\nu = \sum_{i=1}^n n_i \varpi_i$ for some $n_i \gg 0$. Then $x^\nu f = c_\lambda x^\lambda + \sum_{\mu \neq \lambda} c_\mu x^\mu$ where λ is large enough and $c_\lambda \neq 0$. Since $E_\lambda|_{s=0}$ is well-defined, $x^\nu f = c_\lambda E_\lambda|_{s=0} + \sum_{\mu \neq \lambda} c'_\mu x^\mu$. From Lemma 4.24, the dimension of the generalized Y -eigenspace with respect to the eigenvalue $y(\lambda)$ is one. Hence there exists an element $H \in \mathcal{H}_n^s$ such that $Hx^\nu f = E_\lambda|_{s=0}$. \square

4.5.2. Step Irr-2.

Here, we call $(2(n-1)(r-1), \dots, 4(r-1), 2(r-1), 0) \in \mathbb{Z}^n$ the *specific element*.

Using Proposition 4.2, we can easily check that $(2(n-1)(r-1), \dots, 4(r-1), 2(r-1), 0) \leftrightarrow \lambda$ for any large enough λ .

4.5.3. Step Irr-3.

Put

$$\lambda^{m,l} := \left(2(n-1)(r-1), \dots, 2(m+2)(r-1), \right. \\ \left. (m(r-1))^l, 2(m+1)(r-1), (m(r-1))^{m+1-l} \right)$$

for $0 \leq m \leq n-2$ and $0 \leq l \leq m+1$, and

$$\lambda^{n-1,0} := \left((n-1)(r-1), \dots, (n-1)(r-1) \right).$$

Note that the specific element defined above is $\lambda^{0,0}$.

Proposition 4.26. (i) For any $\lambda = \lambda^{m,l}$ with $0 \leq m \leq n-2$ and $0 \leq l \leq m$, or $(m,l) = (n-1,0)$, the non-symmetric Koornwinder polynomial E_λ is well-defined at $s=0$.

(ii) $\lambda^{m,0} \rightarrow \lambda^{m,m}$

(iii) Put $\lambda = \lambda^{m,m+1}$ and $\mu = \lambda^{m+1,0}$ with $0 \leq m \leq n-2$. Then the modified polynomial $\bar{E}_{\lambda,(-1,\mu)} \stackrel{\text{def}}{=} E_\lambda - \frac{\chi_0^*(E_\lambda)}{\chi_0^*(E_\mu)} E_\mu$ is well-defined at $s=0$.

(iv) $\lambda^{m,m} \rightarrow \lambda^{m+1,0}$.

Proof. (i) From the evaluation formula (Proposition 2.11) and recurrence relations (Lemma 2.10), we have $\zeta(\chi_0^*(E_\lambda)) = 0$. On the other hand, for any large enough ν , we have $\zeta(\chi_0(E_\nu^*)) = 0$. Then by putting

$$N = \{\nu \in \mathbb{Z}^n; \nu \text{ is large enough}\},$$

Proposition 4.20 implies that E_λ has no pole at $s=0$.

(ii) Use Proposition 4.2.

(iii) It is easy to see that $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$. From the evaluation formula (Proposition 2.11) and recurrence relations (Lemma 2.10), we have $\zeta(\chi_0^*(E_\lambda)) = -1$ and $\zeta(\chi_0^*(E_\mu)) = 0$.

Thus from Proposition 4.21, we obtain the well-definedness of $\bar{E}_{\lambda,(-1,\mu)}$ at $s=0$.

(iv) Use Proposition 4.18. \square

Corollary 4.27. *We have $\lambda^{0,0} \rightarrow \lambda^{n-1,0} = ((n-1)(r-1), \dots, (n-1)(r-1))$.*

We will finish Step Irr-3.

Proposition 4.28. *We have $((n-1)(r-1), \dots, (n-1)(r-1)) \rightarrow (0, \dots, 0)$.*

Proof. In this proof, for simplicity, we omit the factor $(r-1)$ in each component. For instance, we regard $(\alpha_1, \dots, \alpha_n)$ as $((r-1)\alpha_1, \dots, (r-1)\alpha_n)$.

We denote a data $(\lambda, \{(m_{\lambda\mu}, \mu)\}_\mu)$ by $\lambda + \sum_\mu m_{\lambda\mu}\mu$.

For $m = 1, \dots, n-1$, we will show that

$$\begin{aligned}
 (30) \quad & (m^n) \\
 (31) \quad & \rightarrow (m^{n-1}, 0) - (m^{n-m-1}, (m-1)^m, m) \\
 (32) \quad & \rightarrow ((m-1), m^{n-m-1}, (m-1)^m) \\
 (33) \quad & \rightarrow ((m-1)^n).
 \end{aligned}$$

The step from (30) to (31) is given by iterating the following arrow relations:

$$\begin{aligned}
 & (m^n) \\
 & \leftrightarrow (m-1, m^{n-1}) \\
 & \leftrightarrow (m^{n-3}, m-1, m, m) \\
 & \rightarrow (m^{n-2}, m-1, m) \\
 & \rightarrow (m^{n-1}, m-1) - (m^{n-2}, m-1, m),
 \end{aligned}$$

and for $l = 1, \dots, m-1$

$$\begin{aligned}
 & (m^{n-1}, l) - (m^{n-m+l-1}, (m-1)^{m-l}, m) \\
 & \leftrightarrow (l-1, m^{n-1}) - (m-1, m^{n-m+l-1}, (m-1)^{m-l}) \\
 & \leftrightarrow (m^{n-m+l-3}, l-1, m^{m-l+2}) - (m^{n-m+l-3}, m-1, m^2, (m-1)^{m-l}) \\
 & \rightarrow (m^{n-m+l-2}, l-1, m^{m-l+1}) - (m^{n-m+l-2}, m-1, m, (m-1)^{m-l}) \\
 & \rightarrow \left((m^{n-m+l-1}, l-1, m^{m-l}) - (m^{n-m+l-2}, l-1, m^{m-l+1}) \right) \\
 & \quad - (m-l+1) \times \left((m^{n-m+l-2}, m, m-1, (m-1)^{m-l}) \right. \\
 & \quad \quad \left. - (m^{n-m+l-2}, m-1, m, (m-1)^{m-l}) \right) \\
 & \rightarrow (m^{n-m+l}, l-1, m^{m-l-1}) - (m^{n-m+l-2}, (m-1)^2, m, (m-1)^{m-l-1}) \\
 & \leftrightarrow (m^{n-1}, l-1) - (m^{n-m+l-2}, (m-1)^{m-l+1}, m).
 \end{aligned}$$

The step from (31) to (32) is given as follows:

$$\begin{aligned}
 & (m^{n-1}, 0) - (m^{n-m-1}, (m-1)^m, m) \\
 & \rightarrow (m^{n-m-1}, (m-1)^m, -m) \quad (\text{Proposition 4.19}) \\
 & \leftrightarrow ((m-1), m^{n-m-1}, (m-1)^m).
 \end{aligned}$$

The step from (32) to (33) is given by iterating the following arrow relations: for $l = m, \dots, n-2$, we have

$$\begin{aligned}
& ((m-1), m^{n-l-1}, (m-1)^l) \\
& \leftrightarrow (m^{n-l-3}, (m-1), m, m, (m-1)^l) \\
& \rightarrow (m^{n-l-2}, (m-1), m, (m-1)^l) \\
& \rightarrow (m^{n-l-2}, m, (m-1), (m-1)^l) - (m^{n-l-2}, (m-1), m, (m-1)^l) \\
& \rightarrow (m^{n-l-2}, (m-1), (m-1), m, (m-1)^{l-1}) \quad (\text{Proposition 4.19}) \\
& \leftrightarrow (m^{n-l-2}, (m-1)^{l+1}, m) \\
& \leftrightarrow ((m-1), m^{n-l-2}, (m-1)^{l+1}).
\end{aligned}$$

By repeating from (30) to (33), we obtain the desired statement. \square

We have shown $\lambda^{0,0} \rightarrow (0, \dots, 0)$, and finished the proof of Theorem 4.22.

4.6. The case $t^{k+1}q^{r-1}a^{*2} = 1$. Fix $2n-2 \geq k+1 \geq 0$ and $r-1 \geq 1$. Let $s \in \mathcal{A}$ be an irreducible factor of

$$t^{k+1}q^{r-1}a^{*2} - 1.$$

The main theorem in this subsection is as follows.

Theorem 4.29. *The polynomial representation P_n^s is irreducible.*

Similarly to §4.5, the proof is given by the three steps.

4.6.1. Step Irr-1.

Definition 4.30 (large enough elements). In this subsection, $\lambda \in \mathbb{Z}^n$ is called *large enough* if $\lambda_i \geq r-1$ for any $1 \leq i \leq n$.

Lemma 4.31. *Take any large enough λ . Then there are no $\mu \neq \lambda$ satisfying $y(\mu) = y(\lambda)$ or $y^*(\mu) = y^*(\lambda)$ at $s = 0$. As a corollary, E_λ and E_λ^* have no pole at $s = 0$.*

Proof. Take μ satisfying $y(\mu) = y(\lambda)$ or $y^*(\mu) = y^*(\lambda)$ at $s = 0$. Then there exist integers m_1, \dots, m_n such that $\mu_i = \lambda_i + (r-1)m_i$, $\rho(\mu)_i = \rho(\lambda)_i + m_i$, and $\sigma(\mu)_i = \sigma(\lambda)_i$. However there is no such μ except for $\mu = \lambda$. \square

Similarly to the argument in Lemma 4.25, for any Laurent polynomial $f \in P_n^s$, there exists an element $\lambda \in \mathbb{Z}^n$ such that $\lambda_i \geq r-1$ (for any $1 \leq i \leq n$) and $E_\lambda|_{s=0} \in \mathcal{H}_n^s f$.

4.6.2. Step Irr-2.

Here, we call $(r-1, \dots, r-1) \in \mathbb{Z}^n$ the *specific element*. Using Proposition 4.2, we can easily check that $\lambda \leftrightarrow (r-1, \dots, r-1)$ for any large enough λ .

4.6.3. *Step Irr-3.*

We will show $(r-1, \dots, r-1) \rightarrow (0, \dots, 0)$. For simplicity, for any integer m and natural number ℓ , we denote by m^ℓ the sequence m, \dots, m of ℓ -times repetition.

Lemma 4.32. *Suppose $n-1 \geq k+1 \geq 1$. Put $R := (r-1)^{n-(k+3)}$ and*

$$\begin{aligned} \lambda^{(1)} &= (R, r-1, (-(r-1))^{k+2}), & \mu^{(1)} &= (R, r-1, 0^{k+2}), \\ \lambda^{(2)} &= (R, (-(r-1))^{k+2}, -(r-1)), & \mu^{(2)} &= (R, 0^{k+2}, -(r-1)). \end{aligned}$$

Then for $i = 1, 2$, we have

- (I) $y(\lambda^{(i)})|_{s=0} = y(\mu^{(i)})|_{s=0}$,
- (II) $\zeta(\chi_0^*(E_{\lambda^{(i)}})) = -1$, $\zeta(\chi_0^*(E_{\mu^{(i)}})) = \zeta(\chi_0^*(E_{s_n \lambda^{(1)}})) = \zeta(\chi_0^*(E_{s_{n-1} \mu^{(2)}})) = 0$,
- (III) $E_{s_n \lambda^{(1)}} \rightarrow E_{\mu^{(1)}}$ and $\bar{E}_{\lambda^{(2)}, (-1, \mu^{(2)})} \rightarrow E_{s_{n-1} \mu^{(2)}}$.

Proof. (I) is clear. (II) is from Proposition 2.11 and Lemma 2.10.

(III) Put $N = \{\nu \in \mathbb{Z}^n; \nu \text{ is large enough}\}$. Then from Proposition 4.20 and Proposition 4.21, we see that four polynomials $E_{\mu^{(i)}}$, $\bar{E}_{\lambda^{(i)}, (-1, \mu^{(i)})}$, $E_{s_n \lambda^{(1)}}$, and $E_{s_{n-1} \mu^{(2)}}$ have no pole at $s = 0$. Thus from Proposition 4.18, we have $E_{s_n \lambda^{(1)}} \rightarrow E_{\mu^{(1)}}$ and from Proposition 4.19, we have $\bar{E}_{\lambda^{(2)}, (-1, \mu^{(2)})} \rightarrow E_{s_{n-1} \mu^{(2)}}$. \square

Lemma 4.33. *Let $\lambda^{(1)}$ and $\mu^{(1)}$ as above. Then $\mu^{(1)} \rightarrow (\lambda^{(1)}, \{(-1, \mu^{(1)})\})$.*

Proof. From Theorem 3.14, we have

$$\mu^{(1)} \rightarrow (\lambda^{(1)}, \{(n_\nu, \nu); \nu \in {}^\exists S\}).$$

On the other hand, if $y(\nu)|_{s=0} = y(\lambda^{(1)})|_{s=0}$ for some $\nu \in \mathbb{Z}^n$, then $\nu = \lambda^{(1)}$ or $\nu = \mu^{(1)}$. Since $\zeta\left(\frac{\chi_0^*(E_{\lambda^{(1)}})}{\chi_0^*(E_{\mu^{(1)}})}\right) = -1$, the only possibility of the modification data $\{(n_\nu, \nu); \nu \in S\}$ is

$$\{(n_\nu, \nu); \nu \in S\} = (-1, \mu^{(1)}).$$

\square

Lemma 4.34. *Suppose that $2n - 2 \geq k + 1 > n - 1$. For $i = 3, \dots, 6$, let $\lambda^{(i)}, \lambda^{(i)'}, \mu^{(i)}, \mu^{(i)'}$ be as follows:*

(for odd k),

$$\begin{aligned}\lambda^{(3)} &= ((r-1)^{n-\frac{k+1}{2}-1}, -(r-1), 0, 0^{\frac{k+1}{2}-1}), \\ \lambda^{(3)'} &= ((r-1)^{n-\frac{k+1}{2}-1}, 0, -(r-1), 0^{\frac{k+1}{2}-1}), \\ \mu^{(3)} &= ((r-1)^{n-\frac{k+1}{2}-1}, 0, 0, 0^{\frac{k+1}{2}-1}),\end{aligned}$$

(for even k),

$$\begin{aligned}\lambda^{(4)} &= ((r-1)^{n-\frac{k}{2}-2}, 0, -(r-1), 0^{\frac{k}{2}-1}), \\ \lambda^{(4)'} &= ((r-1)^{n-\frac{k}{2}-2}, 0, 0, -(r-1), 0^{\frac{k}{2}-1}), \\ \mu^{(4)} &= ((r-1)^{n-\frac{k}{2}-2}, -(r-1), 0, 0, 0^{\frac{k}{2}-1}),\end{aligned}$$

for $2k \geq 2l \geq k + 3$,

$$\begin{aligned}\lambda^{(5)} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, 0, -(r-1), 0, 0^{k-l}), \\ \lambda^{(5)'} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, 0, 0, -(r-1), 0^{k-l}), \\ \mu^{(5)} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, -(r-1), 0, 0, 0^{k-l}), \\ \lambda^{(6)} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, 0, -(r-1), 0, 0^{k-l}), \\ \mu^{(6)} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, -(r-1), 0, 0, 0^{k-l}), \\ \mu^{(6)'} &= ((r-1)^{n-l-1}, 0^{2l-k-3}, -(r-1), 0, 0, 0, 0^{k-l}).\end{aligned}$$

Then we have (I) $y(\lambda^{(i)}) = y(\mu^{(i)})$ at $s = 0$,

(II) $\zeta(\chi_0^*(E_{\lambda^{(i)}})) = -1$, $\zeta(\chi_0^*(E_{\mu^{(i)}})) = \zeta(\chi_0^*(E_{\lambda^{(i)'}})) = \zeta(\chi_0^*(E_{\mu^{(6)'}})) = 0$,

(III) $E_{\lambda^{(i)'}} \rightarrow E_{\mu^{(i)}} (i = 3, \dots, 5)$ and $\bar{E}_{\lambda^{(6)'}, (-1, \mu^{(6)})} \rightarrow E_{\mu^{(6)'}}$.

Proof. (I) is clear. (II) is from Proposition 2.11 and Lemma 2.10.

(III) Put $N = \{\nu \in \mathbb{Z}^n; \nu \text{ is large enough}\}$. Then from Proposition 4.20 and Proposition 4.21, we see that four polynomials $E_{\mu^{(i)}}$, $\bar{E}_{\lambda^{(i)'}, (-1, \mu^{(i)})}$, $E_{\lambda^{(i)'}}$, and $E_{\mu^{(6)'}}$ have no pole at $s = 0$. Thus from Proposition 4.18, we have $E_{\lambda^{(i)'}} \rightarrow E_{\mu^{(i)}} (i = 3, \dots, 5)$ and from Proposition 4.19, we have $\bar{E}_{\lambda^{(6)'}, (-1, \mu^{(6)})} \rightarrow E_{\mu^{(6)'}}$. \square

Combining these lemmas, we obtain Step Irr-3.

Proof of Step Irr-3. The case $n - 2 \geq k + 1 \geq 0$. We denote the repetition $(r - 1)^{n-(k+3)}$ by R .

$$\begin{aligned}
(34) \quad & (R, (r - 1)^{k+3}) \\
& \leftrightarrow (R, r - 1, (-(r - 1))^{k+1}, r - 1) \\
(35) \quad & \xrightarrow{\phi_n} (R, r - 1, 0^{k+2}) \\
(36) \quad & \rightarrow (R, r - 1, (-(r - 1))^{k+2}) - (R, r - 1, 0^{k+2}) \\
(37) \quad & \leftrightarrow (R, (-(r - 1))^{k+2}, -(r - 1)) - (R, 0^{k+2}, -(r - 1)) \\
(38) \quad & \xrightarrow{\phi_{n-1}} (R, 0^{k+1}, -(r - 1), 0) \\
& \leftrightarrow (R, 0^{k+1}, 0, 0) \\
& \leftrightarrow (0, \dots, 0).
\end{aligned}$$

The step from (34) to (35) is from Lemma 4.32-(1). The step from (35) to (36) is from Lemma 4.33. The step from (37) to (38) is from Lemma 4.32-(2).

The case $k + 1 = n - 1$. The procedure above stops in the third step.

The case $2n - 2 \geq k + 1 > n - 1$. We have

$$\begin{aligned}
(39) \quad & ((r - 1)^n) \\
& \leftrightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 1}, 0^{\lfloor \frac{k}{2} \rfloor + 1}) \\
(40) \quad & \rightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, 0^{\lfloor \frac{k}{2} \rfloor + 2})
\end{aligned}$$

The step from (39) to (40) is given as follows: If $k + 1$ is even,

$$\begin{aligned}
(39) \quad & \leftrightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, 0^{\frac{k+1}{2}}, -(r - 1)) \\
(41) \quad & \leftrightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, 0, -(r - 1), 0^{\frac{k+1}{2} - 1}) \\
& \rightarrow (40)
\end{aligned}$$

The step from (41) to (40) is from Lemma 4.34-(3). If $k + 1$ is odd,

$$\begin{aligned}
(39) \quad & \leftrightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, 0^{\frac{k}{2} + 1}, -(r - 1)) \\
(42) \quad & \leftrightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, 0, 0, -(r - 1), 0^{\frac{k}{2} - 1}) \\
(43) \quad & \rightarrow ((r - 1)^{n-\lfloor \frac{k}{2} \rfloor - 2}, -(r - 1), 0, 0, 0^{\frac{k}{2} - 1}) \\
& \leftrightarrow (40).
\end{aligned}$$

The step from (42) to (43) is from Lemma 4.34-(4).

We have $(40) \rightarrow (0^n)$ by iterating the following steps: for $2k \geq 2l \geq k+3$

$$\begin{aligned}
& ((r-1)^{n-l}, 0^l) \\
& \leftrightarrow ((r-1)^{n-l-1}, 0^l, -(r-1)) \\
(44) \quad & \leftrightarrow ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, 0, 0, -(r-1), 0^{k-l}) \\
(45) \quad & \rightarrow ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, -(r-1), 0, 0, 0^{k-l}) \\
& \rightarrow ((r-1)^{n-l-1}, 0^{2l-k-3}, 0, 0, -(r-1), 0, 0^{k-l}) \\
(46) \quad & -((r-1)^{n-l-1}, 0^{2l-k-3}, 0, -(r-1), 0, 0, 0^{k-l}) \\
(47) \quad & \rightarrow ((r-1)^{n-l-1}, 0^{2l-k-3}, -(r-1), 0, 0, 0, 0^{k-l}) \\
& \leftrightarrow ((r-1)^{n-l-1}, 0^{l+1}).
\end{aligned}$$

The step from (44) to (45) is from Lemma 4.34-(5). The step from (45) to (46) is given by applying the case (ii) of Proposition 3.12. The step from (46) to (47) is from Lemma 4.34-(6). \square

4.7. The case $t^{n-i}q^{r-1}a^*b^{*\pm 1} = 1$. Fix $n \geq i \geq 1$, $r-1 \geq 1$ and a sign ± 1 . We consider the following specialization of parameters:

$$(48) \quad t^{n-i}q^{r-1}a^*b^{*\pm 1} - 1 = 0.$$

Let $s = s_{\pm} \in \mathcal{A}$ be an irreducible factor of the left hand side.

The purpose of this subsection is to give a unique composition series of the polynomial representation P_n^s , and to give a characterization of the irreducible subspace for the case $s = s_+$.

First we check that each Y -eigenspace is one-dimensional.

Proposition 4.35. *For any $\lambda \in \mathbb{Z}^n$, there is no $\mu \neq \lambda$ such that $y(\lambda) = y(\mu)$. As a corollary, $E_{\lambda}|_{s=0}$ are well-defined.*

Proof. This is a corollary of Lemma 3.2. \square

Theorem 4.36. *The space*

$$V = \text{span}_{\mathbb{K}_s} \{E_{\lambda}|_{s=0}; \lambda_i^+ > r-1, \text{ or } \lambda_i^+ = r-1 \text{ and } \sigma(\lambda)_i = +1\}$$

is the unique irreducible subrepresentation of P_n^s .

Proof. We see that $\zeta(N_j(\lambda)) = \zeta(D_j(\lambda)) = \zeta(c_{j,\lambda}) = 0$ except for the case $j = n$, $\lambda_n = \pm(r-1)$, $\rho(\lambda)_n = \pm(n-i)$. Hence from Proposition 4.2, we have $\lambda \rightarrow s_j \lambda$ except for the case $j = n$, $\lambda_n = \pm(r-1)$, $\rho(\lambda)_n = \pm(n-i)$.

If $j = n$, $\lambda_n = -(r-1)$, $\rho(\lambda)_n = -(n-i)$, then $\zeta(D_n(\lambda)) = \zeta(c_{n,\lambda}) = 0$. Hence $\phi_n E_{\lambda}|_{s=0} = c_{n,\lambda} E_{s_n \lambda}|_{s=0} \neq 0$.

If $j = n$, $\lambda_n = r-1$, $\rho(\lambda)_n = n-i$, then $\zeta(D_n(\lambda)) = 0$ and $\zeta(c_{n,\lambda}) = 1$. Hence $\phi_n E_{\lambda}|_{s=0} = c_{n,\lambda} E_{s_n \lambda}|_{s=0} = 0$.

Therefore the space V and the quotient space P_n^s/V are irreducible. \square

If $s = s_+$, there is a characterization of the irreducible subspace in terms of a vanishing condition.

Lemma 4.37. *Let*

$$S = \{\lambda \in \mathbb{Z}^n; \lambda_i^+ > r-1, \text{ or } \lambda_i^+ = r-1 \text{ and } \sigma(\lambda)_i = +1\}.$$

For any $\lambda \in S$, $\zeta_{s_+=0}(\chi_0^(E_\lambda)) = 1$. For any $\mu \notin S$, $\zeta_{s_+=0}(\chi_0(E_\mu^*)) = 0$.*

Proof. Check the evaluation formula. (Proposition 2.11 and Lemma 2.10). \square

Proposition 4.38 (characterization of V). *We have*

$$V = \{f \in P_n^{s_+}; \chi_\mu^*(f)|_{s_+=0} = 0 \text{ for any } \mu \notin S\}.$$

Proof. Use the duality relation. The proof is similar to that in Proposition 4.15. \square

4.8. The case $t^{n-i}q^{r-1}a^*c^{*\pm 1} = 1$ or $t^{n-i}q^{r-1}a^*d^{*\pm 1} = 1$. Fix $1 \leq i \leq n$, $r-1 \geq 1$, and a sign ± 1 . We consider the following specialization of parameters:

$$(49) \quad t^{n-i}q^{r-1-\theta(\pm 1)}a^*c^{*\pm 1} - 1 = 0, \quad \text{or}$$

$$(50) \quad t^{n-i}q^{r-1-\theta(\pm 1)}a^*d^{*\pm 1} - 1 = 0,$$

where $\theta(+1) = 1$ and $\theta(-1) = 0$. Let $s = s_\pm \in \mathcal{A}$ be an irreducible factor of the left hand side.

The purpose of this subsection is similar to §4.7. We give a unique composition series of the polynomial representation P_n^s , and give a characterization of the irreducible subspace for the case $s = s_+$. Since proofs are also similar, we omit the proofs.

Proposition 4.39. *For any $\lambda \in \mathbb{Z}^n$, there is no $\mu \neq \lambda$ such that $y(\lambda) = y(\mu)$. As a corollary, $E_\lambda|_{s=0}$ are well-defined.*

Proposition 4.40. *The space*

$$V = \text{span}_{\mathbb{K}_s}\{E_\lambda|_{s=0}; \lambda_i^+ > r-1, \text{ or } \lambda_i^+ = r-1 \text{ and } \sigma(\lambda)_i = -1\}$$

is the unique irreducible subrepresentation of P_n^s .

Hereafter, we assume that $s = s_+$.

Lemma 4.41. *Let*

$$S = \{\lambda \in \mathbb{Z}^n; \lambda_i^+ > r-1, \text{ or } \lambda_i^+ = r-1 \text{ and } \sigma(\lambda)_i = -1\}.$$

For any $\lambda \in S$, $\zeta_{s_+=0}(\chi_0^(E_\lambda)) = 1$. For any $\mu \notin S$, $\zeta_{s_+=0}(\chi_0(E_\mu^*)) = 0$.*

Proposition 4.42 (characterization). *We have*

$$V = \{f \in P_n^{s_+}; \chi_\mu^*(f)|_{s_+=0} = 0 \text{ for any } \mu \notin S\}.$$

Remark 4.43. In [vDSt], Equation (3.6), van Diejen and Stokman treated the specialization of parameters of the following form:

$$t_l t_m t^{n-1} q^N = 1 \quad (0 \leq l \neq m \leq 3).$$

The correspondence of their parameters and the present parameters is given by $t_0 = a, t_1 = b, t_2 = c, t_3 = d$. If $l = 0$, then their specializations correspond to our specializations (48), (49), (50) where $i = 1$ and the signs are plus. Proposition 3.7 in [vDSt] shows that certain *symmetric* Koornwinder polynomials for specialized parameters vanish at certain finite grid points. Therefore, the characterization of V in this paper can be considered as a non-symmetric version and a generalization to $i \geq 2$.

4.9. The case $q^{r-1} = 1$. In this subsection, we consider the case (8) with $k + 1 = 0$. That is, suppose that $r - 1 \geq 1$ and let $s \in \mathcal{A}$ be an irreducible factor of

$$q - \omega_{r-1}$$

where ω_{r-1} is a primitive $(r - 1)$ -th root of unity.

The purpose of this subsection is to show that P_n^s is Y -semisimple and to give infinitely-many subrepresentations V_μ of P_n^s which are labelled by partitions μ of length $\leq n$. (We only consider partitions of length $\leq n$ and we omit “of length $\leq n$ ” here, for simplicity.) The inclusion relation $V_\mu \supseteq V_{\mu'}$ holds if and only if $\mu \leq \mu'$ where \leq is the dominance ordering of partitions. Any successive quotient of them is isomorphic to each other.

For any fixed $\lambda \in \mathbb{Z}^n$, there are infinitely-many $\mu \in \mathbb{Z}^n$ such that $y(\lambda)|_{s=0} = y(\mu)|_{s=0}$ since $q|_{s=0}$ is a root of unity. However, we have the following statement.

Proposition 4.44. *For any $\lambda \in \mathbb{Z}^n$, $E_\lambda|_{s=0}$ is well-defined.*

Proof. Recall the recursive construction of a basis of P_n^s in Theorem 3.14 and Proposition 3.12. Since $D_i(\lambda)|_{s=0} \neq 0$ for any $0 \leq i \leq n$ and any $\lambda \in \mathbb{Z}^n$, we see that the modification term $\text{mt}(\lambda) = \emptyset$ and $\bar{E}_\lambda = E_\lambda$. Well-definedness of the modified polynomial \bar{E}_λ at $s = 0$ implies the desired statement. \square

We will introduce a labelling set of subrepresentations.

Definition 4.45 $((r - 1)$ -quotient). For any $\lambda \in \mathbb{Z}^n$, take the indexes $1 \leq i_1, \dots, i_n \leq n$ by

$$(i_1, \dots, i_n) = (|w_\lambda^+(1)|, \dots, |w_\lambda^+(n)|).$$

Put $i_{n+1} = n+1$, $\lambda_{i_{n+1}} = 0$, and $\sigma(\lambda_{i_{n+1}}) = +1$. Take non-negative integers p_m ($1 \leq m \leq n$) by

$$p_m = \left\lfloor \frac{|\lambda_{i_m}| - |\lambda_{i_{m+1}}| - \beta_m}{r-1} \right\rfloor,$$

where $\beta_m = 1$ if $(\sigma(\lambda)_{i_m}, \sigma(\lambda)_{i_{m+1}}) = (+, +)$ and $i_m > i_{m+1}$, or $(\sigma(\lambda)_{i_m}, \sigma(\lambda)_{i_{m+1}}) = (-, -)$ and $i_m > i_{m+1}$, or $(\sigma(\lambda)_{i_m}, \sigma(\lambda)_{i_{m+1}}) = (-, +)$,
and $\beta_m = 0$ otherwise.

Then we call the partition

$$\begin{aligned} \lambda^{\text{quot}} &:= \sum_{i=1}^n p_i \varpi_i \\ &= \left(\sum_{i=1}^n p_i, \sum_{i=2}^n p_i, \dots, \sum_{i=n}^n p_i \right) \end{aligned}$$

the $(r-1)$ -quotient of λ . We define the dominant element $\lambda^{\text{std}} \in \mathbb{Z}^n$ by

$$\lambda^{\text{std}} := ((r-1)\lambda_1^{\text{quot}}, \dots, (r-1)\lambda_n^{\text{quot}}).$$

Note that $(\lambda^{\text{std}})^{\text{quot}} = \lambda^{\text{quot}}$.

For example, let $n = 4$, $r-1 = 3$ and $\lambda = (-3, 0, -9, 13)$. Then $(i_1, i_2, i_3, i_4) = (3, 4, 2, 1)$, $(p_1, p_2, p_3, p_4) = (1, 2, 0, 0)$, 3-quotient of λ is $\lambda^{\text{quot}} = (3, 2, 0, 0)$, and $\lambda^{\text{std}} = (9, 6, 0, 0)$.

For any partition μ , put

$$\begin{aligned} \mathbb{Z}_\mu^n &:= \{ \text{the } (r-1)\text{-quotient of } \lambda \text{ is } \mu \}, \\ \mathbb{Z}_{\geq \mu}^n &:= \bigsqcup_{\mu': \text{partition}, \mu' \geq \mu} \mathbb{Z}_{\mu'}^n. \end{aligned}$$

Then \mathbb{Z}^n is decomposed as follows:

$$\mathbb{Z}^n = \bigsqcup_{\mu: \text{partition}} \mathbb{Z}_\mu^n.$$

Define vector spaces as follows:

$$\begin{aligned} V_\mu &:= \text{span}_{\mathbb{K}_s} \{E_\lambda|_{s=0}; \lambda \in \mathbb{Z}_\mu^n\}, \\ V_{\geq \mu} &:= \text{span}_{\mathbb{K}_s} \{E_\lambda|_{s=0}; \lambda \in \mathbb{Z}_{\geq \mu}^n\}. \end{aligned}$$

Now we give the main theorem of this subsection.

Theorem 4.46. (i) For any partition μ and any $\lambda \in \mathbb{Z}_\mu^n$, $\mathcal{H}_n^s E_\lambda|_{s=0}$ coincides with $V_{\geq \mu}$. That is, $V_{\geq \mu}$ is a subrepresentation of P_n^s .

(ii) For any partitions μ and ν , we have $V_{\geq \mu} \subseteq V_{\geq \nu}$ if and only if $\mu \geq \nu$.

(iii) For any partition μ ,

$$V'_\mu := V_{\geq \mu} \Big/ \left(\sum_{\mu' > \mu} V_{\geq \mu'} \right)$$

is finite dimensional and irreducible. As \mathbb{K}_s -vector spaces, $V'_\mu \cong V_\mu$. For any $\lambda \neq \lambda' \in \mathbb{Z}_\mu^n$, we have $y(\lambda)|_{s=0} \neq y(\lambda')|_{s=0}$. That is, the dimension of each Y -eigenspace in V'_μ is 1.

(iv) For any partition μ and ν , we have $V'_\mu \cong V'_\nu$ as \mathcal{H}_n^s -modules.

Proof. We prove the desired statements by using lemmas given after the proof.

(i) We have

$$\begin{aligned} \mathcal{H}_n^s E_\lambda|_{s=0} &= \sum_{\ell \geq 0, 0 \leq i_1, \dots, i_\ell \leq n} \mathbb{K}_s(\phi_{i_1} \cdots \phi_{i_\ell} E_\lambda)|_{s=0} \\ &= \sum_{\nu^{\text{quot}} \geq \mu} \mathbb{K}_s E_\nu|_{s=0} \quad (\text{from Lemma 4.47}) \\ &= V_{\geq \mu}. \end{aligned}$$

(ii) We see that

$$\begin{aligned} V_{\geq \mu} &\subseteq V_{\geq \nu} \\ \Leftrightarrow \mathbb{Z}_{\geq \mu}^n &\subseteq \mathbb{Z}_{\geq \nu}^n \\ \Leftrightarrow \mu &\geq \nu. \end{aligned}$$

(iii) It is clear that $V'_\mu \cong V_\mu$ as \mathbb{K}_s -vector spaces. We see that V'_μ is finite dimensional because \mathbb{Z}_μ^n is a finite set. Suppose that $y(\lambda)|_{s=0} = y(\lambda')|_{s=0}$ for some $\lambda, \lambda' \in \mathbb{Z}_\mu^n$. Then $\rho(\lambda) = \rho(\lambda')$, $\sigma(\lambda) = \sigma(\lambda')$, and $\lambda_i \equiv \lambda'_i \pmod{r-1}$ for any $1 \leq i \leq n$. Since λ and λ' are elements in \mathbb{Z}_μ^n , λ^{quot} should be equal to λ'^{quot} . Thus $\lambda_i = \lambda'_i$ for any $1 \leq i \leq n$. Therefore, the irreducibility of V'_μ follows from (i), (ii), and Lemma 4.49.

(iv) We show that $V'_\mu \cong V'_\nu$ for any partition μ and $\nu = (0, \dots, 0)$. For any $\lambda \in \mathbb{Z}_\mu^n$, we define $\lambda' \in \mathbb{Z}^n$ as follows: let i_1, \dots, i_n and p_1, \dots, p_n be that of Definition 4.45. Put

$$(51) \quad \lambda'_{i_m} := \lambda_{i_m} - \text{sgn}(\lambda_{i_m})(r-1) \left(\sum_{j=m}^n p_j \right) \quad (1 \leq m \leq n).$$

(For example, if $\lambda = (-3, 0, -9, 13)$, then $\lambda' = (-3, 0, -3, 4)$.) We see that λ' is an element in $\mathbb{Z}_{(0, \dots, 0)}^n$, and the map $\lambda \mapsto \lambda'$ gives the isomorphism $\mathbb{Z}_\mu^n \cong \mathbb{Z}_{(0, \dots, 0)}^n$ as finite sets. By the definition of λ' (51), we have $y(\lambda)|_{s=0} = y(\lambda')|_{s=0}$. Therefore, the actions of Y_1, \dots, Y_n and ϕ_0, \dots, ϕ_n on E_λ coincides with those on $E_{\lambda'}$, and the map $E_\lambda \mapsto E_{\lambda'}$ extends to an isomorphism of \mathcal{H}_n^s -modules. \square

The following two lemmas are tools for the proof of Theorem 4.46.

Lemma 4.47. Fix $0 \leq i \leq n$ and $\lambda \in \mathbb{Z}_\mu^n$ such that $s_i \cdot \lambda \neq \lambda$. Then we have:

(i) $c_{i,s_i \cdot \lambda}|_{s=0} = 0$ if and only if $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}} + \varpi_j$ for some j .

(ii) $c_{i,\lambda}|_{s=0} = 0$ if and only if $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}} - \varpi_j$ for some j .

(iii) $c_{i,\lambda}|_{s=0} \neq 0$ and $c_{i,s_i \cdot \lambda}|_{s=0} \neq 0$ if and only if $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}}$.

(The case $c_{i,\lambda}|_{s=0} = 0$ and $c_{i,s_i \cdot \lambda}|_{s=0} = 0$ does not occur by the definition of $c_{i,\lambda}$ (see Proposition 2.5).)

Corollary 4.48. We have the following facts. (Recall the arrow relation “ \rightarrow ” defined in Definition 4.1.) In the case (i), we have $\lambda \rightarrow s_i \cdot \lambda$ and $(\phi_i E_{s_i \cdot \lambda})|_{s=0} = 0$. In the case (ii), we have $s_i \cdot \lambda \rightarrow \lambda$ and $(\phi_i E_\lambda)|_{s=0} = 0$. In the case (iii), we have $\lambda \leftrightarrow s_i \cdot \lambda$.

For example, if $n = 4$, $r - 1 = 3$ and $\lambda = (-3, 0, -9, 13)$, then

$$\begin{aligned} s_0 &: (-3, 0, -9, 13) \leftrightarrow (2, 0, -9, 13), \\ s_1 &: (-3, 0, -9, 13) \rightarrow (0, -3, -9, 13), \\ s_2 &: (-3, 0, -9, 13) \leftrightarrow (-3, -9, 0, 13), \\ s_3 &: (-3, 0, -9, 13) \leftrightarrow (-3, 0, 13, -9), \\ s_4 &: (-3, 0, -9, 13) \leftrightarrow (-3, 0, -9, -13). \end{aligned}$$

3-quotient of $(0, -3, -9, 13)$ is $(4, 3, 1, 0)$, which is equal to $(3, 2, 0, 0) + (1, 1, 1, 0)$. 3-quotient of the other elements are equal to $(3, 2, 0, 0)$.

Proof of Lemma 4.47. First we give a proof for (i). (The proof for (ii) is given by switching $s_i \cdot \lambda$ and λ .)

Recall the definition of $c_{i,\lambda}$ (see Proposition 2.5). Suppose that $c_{i,\lambda}|_{s=0} \neq 0$ and $c_{i,s_i \cdot \lambda}|_{s=0} = 0$. Then $\langle s_i \cdot \lambda, \alpha_i \rangle > 0$ and $1 \leq i \leq n$. Moreover, the condition (I) or (II) should be satisfied:

(I) $1 \leq i \leq n - 1$, $(\sigma(s_i \cdot \lambda)_i, \sigma(s_i \cdot \lambda)_{i+1}) = (+1, +1)$ or $(-1, -1)$, $\rho(s_i \cdot \lambda)_i - \rho(s_i \cdot \lambda)_{i+1} = 1$, and $(s_i \cdot \lambda)_i - (s_i \cdot \lambda)_{i+1} = (r - 1)m$ for some $m > 0$.

(II) $i = n$, $\sigma(s_i \cdot \lambda)_i = +1$, $\rho(s_i \cdot \lambda)_i = 0$, and $(s_i \cdot \lambda)_i = (r - 1)m$ for some $m > 0$.

For each case (I) or (II), by the definition of λ^{quot} , we have $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}} + \varpi_j$ for some j .

Conversely, if $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}} + \varpi_j$ for some j , then (I) or (II) occur, and we have that $c_{i,\lambda}|_{s=0} \neq 0$ and $c_{i,s_i \cdot \lambda}|_{s=0} = 0$.

We show (iii). By the definition of λ^{quot} , if $(s_i \cdot \lambda)^{\text{quot}} \neq \lambda^{\text{quot}} \pm \varpi_j$, then $(s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}}$. Since we have proved (i) and (ii), we obtain that $c_{i,\lambda}|_{s=0} \neq 0$ and $c_{i,s_i \cdot \lambda}|_{s=0} \neq 0$. \square

Lemma 4.49. For any $\lambda \in \mathbb{Z}_\mu^n$, we have $\lambda \leftrightarrow \lambda^{\text{std}}$.

Proof. Fix $\lambda \in \mathbb{Z}_\mu^n$. From the previous lemma, if $\lambda \neq s_i \cdot \lambda$ then

$$s_i \cdot \lambda \leftrightarrow \lambda \Leftrightarrow (s_i \cdot \lambda)^{\text{quot}} = \lambda^{\text{quot}}.$$

Hence we will show $\lambda \leftrightarrow \lambda^{\text{std}}$ by applying simple reflections on λ with preserving their $(r - 1)$ -quotients. Such a procedure is realized as follows:

(Step 1) In this step, we will change $\lambda \in \mathbb{Z}^n$ to an element in $\mathbb{Z}_{\geq 0}^n$ as follows. Divide the set of indexes $\{1, \dots, n\}$ into two pieces $\{i_1 < \dots < i_\ell\}$ such that $\lambda_{i_m} < 0$ ($1 \leq \forall i \leq \ell$), and $\{j_1 < \dots < j_{n-\ell}\}$ such that $\lambda_{j_m} \geq 0$ ($1 \leq \forall i \leq n - \ell$). Then putting

$$\lambda' := (-\lambda_{i_\ell} - 1, \dots, -\lambda_{i_1} - 1, \lambda_{j_1}, \dots, \lambda_{j_{n-\ell}}),$$

we have $\lambda'^{\text{quot}} = \lambda^{\text{quot}}$ and $\lambda' \leftrightarrow \lambda$. (Indeed, for a shortest element $s_{k_L} \cdots s_{k_1} \in W$ such that $\lambda' = s_{k_L} \cdots s_{k_1} \cdot \lambda$, the $(r-1)$ -quotients of λ , $s_{k_1} \cdot \lambda$, $s_{k_2} s_{k_1} \cdot \lambda$, \dots , $s_{k_L} \cdots s_{k_1} \cdot \lambda$ are equal.)

Note that $\lambda' \in \mathbb{Z}_{\geq 0}^n$. Hereafter we assume that $\lambda \in \mathbb{Z}_{\geq 0}^n$.

(Step 2) In this step, we will reduce $\lambda \in \mathbb{Z}_{\geq 0}^n$ to a dominant element λ'' as follows. Take the index

$$(i_1, \dots, i_n) = (|w_\lambda^+(1)|, \dots, |w_\lambda^+(n)|).$$

Suppose that $i_{\ell+1} = \ell + 1, i_{\ell+2} = \ell + 2, \dots, i_n = n$. Then for any j satisfying $i_\ell + 1 \leq j \leq \ell$, we see that $\lambda_{i_\ell} < \lambda_j$. Hence by putting

$$\lambda' := (\lambda_{i_{\ell+1}} - 1, \dots, \lambda_\ell - 1, \lambda_1, \lambda_2, \dots, \lambda_{i_\ell}, \lambda_{i_{\ell+1}}, \dots, \lambda_{i_n}),$$

we have $\lambda'^{\text{quot}} = \lambda^{\text{quot}}$ and $\lambda' \leftrightarrow \lambda$. (The reason is similar to that in Step 1.) Moreover new indexes

$$(i'_1, \dots, i'_n) := (|w_{\lambda'}^+(1)|, \dots, |w_{\lambda'}^+(n)|)$$

satisfy $i'_\ell = \ell, i'_{\ell+1} = \ell + 1, \dots, i'_n = n$. Therefore by repeating this inductively, we obtain a dominant element λ'' such that $\lambda''^{\text{quot}} = \lambda^{\text{quot}}$ and $\lambda'' \leftrightarrow \lambda$.

Hereafter we assume that λ is dominant.

(Step 3) In this step, we will reduce a dominant element $\lambda \in \mathbb{Z}^n$ to λ^{std} as follows. Suppose that $\lambda_i = \lambda_i^{\text{std}}$ for any $\ell + 1 \leq i \leq n$. By the definition of λ^{std} , we see $\lambda_\ell \geq \lambda_\ell^{\text{std}}$. If $\lambda_\ell > \lambda_\ell^{\text{std}}$, then by putting

$$\lambda' := (\lambda_1 - 1, \dots, \lambda_\ell - 1, \lambda_{\ell+1}, \dots, \lambda_n),$$

we have $\lambda'^{\text{quot}} = \lambda^{\text{quot}}$ and $\lambda' \leftrightarrow \lambda$. (The reason is similar to that in (i).) Therefore by repeating this inductively, we obtain the desired element λ^{std} and it satisfies $(\lambda^{\text{std}})^{\text{quot}} = \lambda^{\text{quot}}$ and $\lambda^{\text{std}} \leftrightarrow \lambda$. \square

For example, let $n = 4$, $r - 1 = 3$ and $\lambda = (-3, 0, -9, 13)$. Then from Step 1, we obtain $\lambda \leftrightarrow (8, 2, 0, 13)$. From Step 2, we obtain $(8, 2, 0, 13) \leftrightarrow (12, 8, 2, 0)$. From Step 3, we obtain $(12, 8, 2, 0) \leftrightarrow (9, 6, 0, 0) = \lambda^{\text{std}}$. 3-quotient of these elements is $(3, 2, 0, 0)$.

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